The Wonderful and Notorious "TWIN PARADOX" of Special Relativity

by Dwight E. Neuenschwander, Professor Emeritus, Southern Nazarene University

When we first hear of it, the "twin paradox" of special relativity can be a real puzzler, and it can remain unsettling even after one has passed a special relativity course. This article describes the situation that creates the apparent paradox, identifies the implicit but incorrect assumption that makes the paradox seem serious, and offers a resolution.[1] I assume the reader is familiar with a few basic principles of special relativity, including the invariance of the speed of light in a vacuum, spacetime diagrams, proper time, time dilation, and the invariance of the spacetime interval.

I'll borrow a vivid mental picture from Taylor and Wheeler's *Spacetime Physics* by imagining two inertial reference frames: the lab frame that uses coordinates (t, x), and the coasting rocket frame that employs coordinates (t', x'). Let the coasting rocket frame move with constant velocity v relative to the lab frame. The coasting rocket's world line from event A to some event B, as plotted in the spacetime diagram of the lab frame, is shown in Fig. 1. To record times, arrays of clocks synchronized within each frame are set to read zero at event A, where the origins of both coordinate systems coincide.[2] It is crucial to realize that the world line AB of the coasting rocket forms the t'-axis of the coasting rocket frame's coordinate system. Because the speed of light c is invariant, $c = \Delta x/\Delta t = \Delta x'/\Delta t'$, it follows that the world line of a light ray emitted from event A lies halfway between the space and time axes when those axes are measured in the same units. In this discussion, time will be measured in years and distances in light-years. Therefore, when we project the rocket's (t', x') axes onto the (t, x) axes, the world line of the light ray is halfway between the t' and the x' axes, and halfway between the t and x axes. Notice that event B in the rocket frame is simultaneous with event C in the lab frame, as shown in Fig. 1. This is so because the line BC is a spacelike interval in the rocket frame—line BC is parallel to the x'-axis in Fig. 1. But for the lab observer, event C is simultaneous with other events, such as event E in the coasting rocket frame.

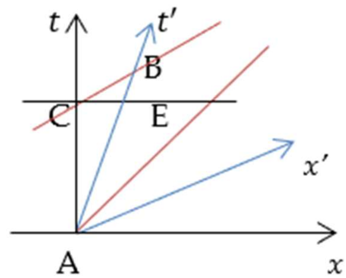

Figure 1. The world line AB of the coasting rocket as mapped in the lab frame's spacetime diagram. The red line that is halfway between the (t, x) and (t', x') axes represents the world line of a beam of light that was emitted from event A. Lines that are parallel to their respective x and x' axes describe spacelike intervals, and events on these lines are simultaneous in their respective frames.

The Paradox—And Its Resolution

A set of twins, Bonnie and Clyde, is born on Earth (the lab frame). They live on Earth for 20 years, at which time Clyde embarks on a journey in his spaceship (the coasting rocket frame) to the star Zeta that, in the Earth's reference frame, is three light-years (3 c-yr) from Earth. Clyde promises to return to Earth as soon as he reaches Zeta and to do so at the same speed as the outbound trip. Suppose this speed is $v = \frac{3}{5}c$ relative to Bonnie. Therefore

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{5}{4}. \tag{1}$$

Let the Earth-lab frame be an inertial reference frame,[3] and let the lab's x-axis be the line from Earth to Zeta. Let Clyde's driver's seat define the coasting rocket frame's origin, $x' = 0$, with the x'-axis parallel to the x-axis.

Event A in Figs. 1 and 2(a) denotes the event where Clyde departs from the Earth. For event A, we have, respectively, the lab frame and coasting rocket frame coordinates $(t_A, x_A) = (t'_A, x'_A) = (0{,}0)$. Let event Z denote the event where Clyde arrives at star Zeta. Event Z occurs in Bonnie's frame at $x_Z = 3\ c$ yr and $t_Z = \Delta x/v = 3\ c\ \mathrm{yr}/(3c/5) = 5$ yr. Meanwhile, $x'_Z = 0$, because in Clyde's coordinate system, the star Zeta came to him, arriving at the rocket frame's origin. Clyde measures proper time[4] between events A and Z, so the time dilation relation $t_Z = \gamma t'_Z$ becomes $5\ \mathrm{yr} = (5/4)\, t'_Z$, which gives $t'_Z = 4$ yr. The upshot is that between events A and Z, Bonnie ages five years according to her clocks, and Clyde ages four years according to his wristwatch. On the return trip, because the distances and speeds are the same as on the outbound trip, when Clyde returns to Bonnie at the reunion event R, Bonnie will be 30 years old and Clyde's age will be 28 years. So far, so good.

The apparent paradox arises when all of these events are examined from Clyde's reference frame. In the coasting rocket frame, in mapping events from A to Z, Clyde sees the Earth and Bonnie receding from him, and Zeta approaching with velocity $-3c/5$. Then from event Z to the reunion event R, Clyde observes Zeta receding and Bonnie approaching him at velocity $+3c/5$. One might be forgiven for assuming that upon Bonnie's return to Clyde, as observed by Clyde, she will be 28 and Clyde will be 30 years old. Hence the paradox: How can Clyde and Bonnie be both 28 and 30 years old?

The motion just described seems at first glance to be symmetrical, as shown by comparing Figs. 2(a) and 2(b)–shouldn't Clyde's motion as mapped in Bonnie's (t, x) spacetime diagram [Fig. 2(a)] be symmetrical with Bonnie's motion as mapped in Clyde's (t', x') spacetime diagram [Fig. 2(b)]?

As in all paradoxes that arise in special relativity, some unconscious assumption is made that contradicts the theory's postulates. Special relativity applies to inertial reference frames only—frames that undergo no acceleration. In the twin paradox, Clyde undergoes an acceleration. Relative to Bonnie, at event Z, his velocity changes from $+3c/5$ to $-3c/5$. Throughout all of the events cited, Bonnie never accelerates. At event Z, Clyde jumps from the rocket frame, with its coordinates (t', x') that moves with velocity $+3\,c/5$ relative to the Earth frame, and boards the return rocket frame with coordinates (t'', x'') that moves with velocity $-3c/5$ relative to the Earth frame.[5] Figure 2(b) is misleading because it suggests that Bonnie undergoes an acceleration when, in fact, she does no such thing. It is Clyde's acceleration at event Z that makes Bonnie's world line reverse direction in Fig. 2(b).

Referring to Fig. 3 now, a few noteworthy events occur simultaneously with event Z. Figure 3 maps the spacetimes of the outgoing coasting rocket frame's (t', x') axes and the return rocket frame's (t'', x'') axes, both projected onto the Earth-lab frame's

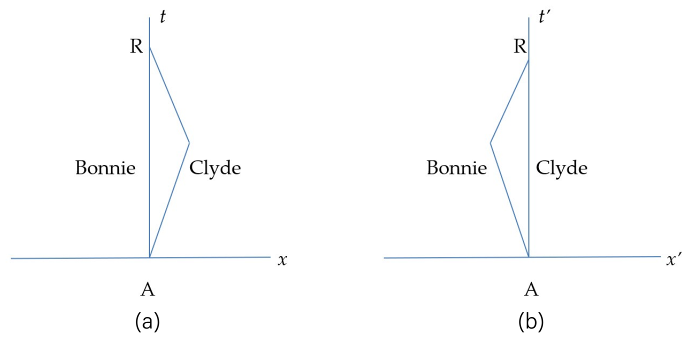

Figure 2. (a) The motion of Bonnie and Clyde as plotted on Bonnie's earth frame spacetime axes from event A to event R. (b) The motion of Bonnie and Clyde as plotted on Clyde's coasting rocket frame spacetime axes. Figure 2(b) motivates a false assumption, giving rise to the paradox.

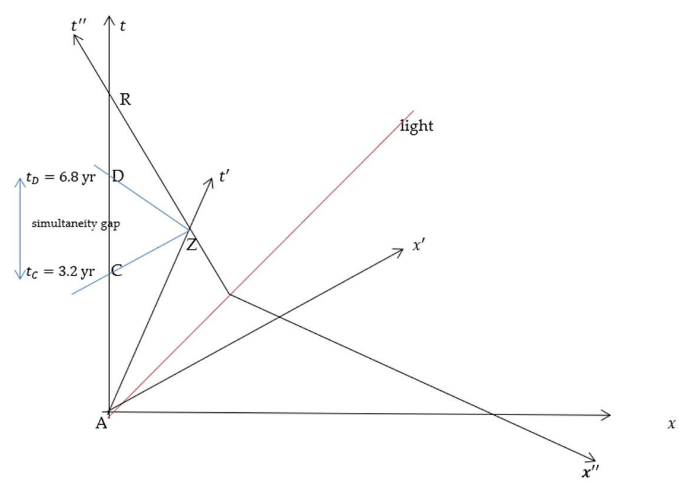

Figure 3. The earth-lab frame (t, x), coasting rocket frame (t', x') and return rocket frame (t'', x'') coordinate axes. Bonnie's world line goes from A to R along the t-axis in the lab frame. Clyde's world line goes from A to Z in the coasting rocket frame, then from Z to R in the return rocket frame. In the coasting rocket frame, events Z and C are simultaneous (line CZ is parallel to the x′-axis). In the return rocket frame, events Z and D are simultaneous (line ZD is parallel to the x″ axis). The time between events C and D is the simultaneity gap. (The coasting rocket origin is placed on the red world line of light to show that the angles between it and the t″ and x″ axes are the same.)

(t, x) axes. Recall that event Z occurs at $t_Z = 5\,\mathrm{yr}$ in Bonnie's frame but in Clyde's frame, $t'_Z = 4\,\mathrm{yr}$. Furthermore, in Clyde's frame event Z is simultaneous with an event C that occurs in Bonnie's frame. We can find Bonnie's clock time for event C using time dilation: When Clyde's clock reads $t'_Z = 4\,\mathrm{yr}$, according to Bonnie, who reads proper time between events A and C (because both occur at $x = 0$), her clocks record $t_C = 4\,\mathrm{yr}/\gamma = 3.2\,\mathrm{yr}$.

In addition, at event Z Clyde jumps from the coasting rocket frame to the return rocket frame. Within the return rocket frame, event Z is also simultaneous with event D on Bonnie's world line in the lab frame! When Clyde jumps from the rocket frame to the return rocket frame at event Z, the time interval between events C and D on Bonnie's world line introduces a simultaneity gap. Since event C occurred 3.2 years after event A according to Bonnie's clocks, event D in Bonnie's frame occurs 3.2 years before the reunion at event R, $t_D = 10 - 3.2 = 6.8$ years. The magnitude of the simultaneity gap as measured by Bonnie's clocks is $t_D - t_C = 6.8 - 3.2 = 3.6$ years.

Meanwhile, using data available to him aboard his rocket frame (presumably Clyde knew the 3 c-yr distance between Earth and Zeta before he left Earth), Clyde knows that when he rejoins Bonnie at the reunion event R, she will have aged 10 years according to her clocks. To see how he knows this, apply the invariance of the spacetime interval,

$$c^2(\Delta t')^2 - (\Delta x')^2 = c^2(\Delta t)^2 - (\Delta x)^2, \qquad (2)$$

to the first half of the journey, from event A to event Z. Inserting A-to-Z data, Eq. (2) can be solved for Bonnie's time Δt:

$$c^2(4\,\mathrm{yr})^2 - (0)^2 = c^2(\Delta t)^2 - (3\,c\,\mathrm{yr})^2, \qquad (3)$$

which gives $\Delta t = 5$ yr halfway through, so that Clyde can predict that at event R, Bonnie will have aged 10 years according to her calendar.

At the reunion event R, Bonnie will have celebrated ten birthdays since event A and Clyde will have celebrated eight birthdays since event A. The "twin paradox" seems paradoxical because the scenario's description implicitly assumes that both Bonnie and Clyde are always in their respective inertial reference frame for all events between A and R. Bonnie does indeed remain in one and only one inertial lab frame all the way from A to R, but Clyde's transition from the coasting rocket frame to the return rocket frame disqualifies him from being in the same inertial frame from A to R. In addition, the simultaneity gap illustrates the robustness of Albert Einstein's original thought experiment about the relativity of time as approached through the noninvariance of simultaneity.[6]

Seeing them reunited at event R, let us toast Bonnie and Clyde on their reunion after Clyde's voyage and raise another 18 toasts to their ten and eight birthdays, respectively!

Endnotes

1. Practically every textbook on special relativity describes the twin paradox. See, for example, John Brehm and William Mullin's *Introduction to the Structure of Matter* (Wiley, 1988)—I have borrowed their numerical parameters because they result in especially simple calculations; Edwin Taylor and John A. Wheeler's *Spacetime Physics* (Freeman, 1966, 1992); Anthony French's *Special Relativity* (Norton, 1968); and Paul Tipler and Ralph Llewellyn's *Modern Physics*, 5th ed. (Freeman, 1988); among many others.
2. The clocks at different locations within a given inertial frame can be synchronized as follows: Set a clock reading zero at the origin, clocks at $x = \pm 1\,\mathrm{m}$ reading $t = 1\mathrm{m}/c$, and so on, but do not let the clocks run yet. When all necessary clocks are arranged, send a light pulse from the origin, and when that light reaches a clock, it begins running. In this way, all the clocks distributed across the *x*-axis will be synchronized. Events throughout the reference frame have their time and position coordinates recorded locally, and a sequence of events can be traced globally from this data.
3. To a good approximation, the v^2/r accelerations due to Earth's spin on its axis, its orbit about the Sun, and the solar system's orbit around the galactic center are negligible in this application.
4. Recall that the proper time between two events is the time interval between them as measured in the reference frame where the two events occur at the same place. In other words, if $\Delta x = 0$ then Δt is the proper time.
5. These notes do not consider Clyde's initial acceleration that got him up to speed at event A or his deceleration following the reunion at event R—the acceleration at event Z is sufficient to make evident the role of acceleration in resolving the paradox. But those initial and final accelerations can be rendered irrelevant anyway by having Clyde get a running start before event A and coming to stop after event R. Nor do we need to consider the rate of change of Clyde's acceleration (his "jerk") at event Z to make the point that it's the existence of the acceleration itself whose neglect makes the adventures of the twins seem, at first glance, paradoxical.
6. Recall Einstein's thought experiment about the train being struck by lightning on both ends [see Albert Einstein, *Relativity: The Special and General Theory* (Crown, 1961), pp. 25-27]. Einstein showed that the reception of the two flashes of light, if simultaneous to the ground-based observer, is not simultaneous for an observer riding on the train.