. . . *of Time and the Sun*

An experimental encounter with the equation of time

By Richard M. Sutton

ON the floor of my office at Haverford College there is painted in several colors a precision sun dial which, after the manner of Topsy, "just growed". It all started in a casual way, but as it has developed for more than two years, it has become increasingly interesting. From it, my students and I have learned much about the *equation of time* and the need for the *mean solar second*. The inappropriateness of the sun as a primary time keeper, despite our constant dependence on its regularity, is clearly disclosed. The method of approach has been that of the natural philosopher who seeks a more complete understanding of the sun's slightly irregular apparent motions. The tools employed are mostly those that have been available through the centuries. With the exception of a pocket watch or an electric clock frequently checked against time signals from the Bureau of Standards' Station WWV, the tools are like those available to the ancient Egyptians or Mayas, and they compete in accuracy with those used by Tycho Brahe.

As in reading a novel it takes the edge off the story if one peeks to see how the story ends, or in working a problem one looks at the answer too soon, so in learning the habits of the sun "it has not been cricket" to consult an almanac too early or too often. Rather, my purpose has been to make the sun unfold its habits and tell its own story. Starting from the inescapable knowledge of the sun's daily rising and setting and its annual trip northward and southward, and from rumors that it usually is not on the meridian at noon, it was exciting to see its apparent motion revealed experimentally. And as study of its habits has continued, it has been rewarding to discover many interesting by-products not initially sought.

Dr Sutton is Associate Director for Science and Development in the Management Development Program at Case Institute of Technology. He joined the Case staff several months ago after more than two decades of teaching physics at Haverford College. From 1944 until his resignation last year he was Chairman of the Haverford Department of Physics.

The Method

IN a window of my office with its south exposure, I have placed a fixed circular hole 1 cm in diameter in an opaque sheet of aluminum at a height of 140 cm above the floor. Sunlight passing through this hole casts an oval spot of light on the floor. This spot moves eastward as the sun apparently progresses westward. It moves northward from June 21 to December 21 as day by day the sun moves southward (see cover). On bright days, the center of this moving spot of light may be marked precisely at any time with an error not exceeding 1 mm. This is done by moving a flat plate or card on the floor until the bright oval is bounded by a rectangle of appropriate size. Then the sharp point of an awl or icepick is inserted through a hole centrally situated to mark the floor.

At first, only the location of the spot at civil noon, Eastern standard time, was punched on successive days. A small tack was driven into the floor for each day. Soon the Figure Eight of an elongated *analemma* began to develop on the floor. Little by little, the pattern was extended by observation and interpolation to show the diurnal paths of the spot on successive days, and its location at five-minute intervals. Now the graph of interlacing analemmas covers the whole portion of the floor reached by sunlight coming through the fixed hole, roughly from 10 A.M. to 1 P.M., from December to June and return. In the interest of doing some other work, it is lucky for me that the sun doesn't shine in more hours each day! It soon becomes apparent that one may interpolate on the dial with confidence, once its habits are known, and a few cloudy days need not cause significant errors. By taking observations every fifteen or twenty minutes, one may still obtain all essential details to a high degree of precision.

The analemma, so often shown on globes of the earth as a Figure Eight in the Pacific Ocean off the coast of South and Central America, is in effect the graph of the "equation of time" curve vs. the declination of the

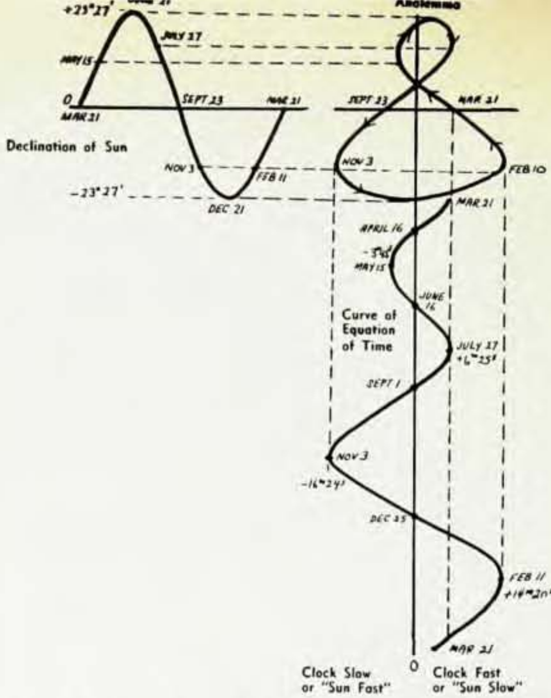

Fig. 1. Lissajous' Figure of Analemma: y = Sun's Declination; x = Equation of Time.

sun (Fig. 1). This is not unlike a Lissajous' Figure Eight for reasons to be discussed, since the motion on the y-axis due to changing declination of the sun as it appears to move north and south has a twelve month period, and the motion on the x-axis is an unsymmetric motion having a large component with six-month period and a smaller component with twelve-month period.

From trigonometric tables and the known longitude and latitude of Haverford, a meridian line and a line showing the path of the spot when the sun is on the celestial equator have been added. There is a simple but excellent way to tell just when the sun is *on* the meridian (or on any other chosen point). If one sets up a fixed card on the floor, inclined so that the shaft of light from the hole falls perpendicularly upon it, one obtains a bright circle on this card as a pinhole image of the sun. It takes approximately two minutes for this circle to advance eastward by the distance of its own diameter. If a circle is drawn on this card and a hole punched through its center, light passing through this small hole makes a very small spot of light on the floor. As the sun's image advances across the stationary card, *this spot remains fixed in position.* Hence, by directing this small spot upon the meridian line before the sun is centered on the meridian, one needs only to watch the sun's image advance and note the time when it is centered within the fixed circle. This decision may be made with an error not to exceed one or two seconds. Of course, this does not match in accuracy the precision of telescopic observation, but remember that this sun dial has been developed much in the manner of pre-telescopic astronomy. No lens is used in the hole. As the sun subtends an angle of approximately 0.55 degree as seen from the earth, the winter image formed at a distance of 3.12 meters from the hole is about 3 cm in diameter, whereas the summer image is reduced to about 1.4 cm at the time when the sun is highest in the sky and the spot on the floor is only 1.5 m from the hole.

The Findings

THE analemma drawn for the civil noon positions of the sun around the year is of impressive extent. For a hole 140 cm above the floor, the June 21 point when the sun's elevation is roughly 73.5 degrees at Haverford is only 40 cm from the window; but for December 21 when this angle of elevation has dropped to 26.5 degrees, the foot of the curve is some 280 cm from the window. At its broadest point, when the sun is at declination 15° S (corresponding to November 3 and February 10) the large loop of the "8" is 32 cm across, corresponding to a time difference of more than 30 minutes. Thus, for a period of about 10 days, the noon curve for November overlaps the 12:30 curve for February, and the noon curve for February crosses the 11:30 curve for November. Such large variations show why an uncorrected sun dial is often in serious error and not a good time keeper.

The diurnal paths made by the spot are almost straight lines near the equinoxes, but are distinctly curved near the solstices. They closely approximate hyperbolas for reasons to be discussed shortly. The diurnal paths at the solstices form a beautiful envelope for all the five-minute analemmas.

The four times each year when the sun is on or close to the meridian at civil noon show clearly where the analemma crosses the meridian line. For Haverford, at longitude 75° 18′ W, these days are April 20, June 7, September 5, and December 23.[1] The longest days between successive crossings of the meridian come in late December when sun days are about 30 seconds longer than 24 hours. The shortest days, in March and September, are not so immediately apparent from the analemma. Thus, near the Autumnal equinox, sun days are about 21 seconds less than 24 hours. The reasons for these variations will be discussed shortly.

Now that the pattern has been developed, the sun dial can be used as a clock on which it is possible to read time on clear days with an error seldom as great as 10 seconds. The average error in some fifty test cases was found to be only 5 seconds! This corresponds to an error in locating the center of the spot of approximately 1 mm. When electric service was restored four days after Hurricane Hazel in October, 1954, I reset my electric clock by the sun dial. Subsequent check of the clock against time signals from Station WWV showed an error less than 3 seconds. Measurement of the sun's declination is possible to a somewhat less degree of accuracy. Comparisons of measured values against almanac values show errors averaging about 3 minutes of arc. Reduction of this error should be readily possible by eliminating several recognized

[1] For any geographical location more than 4° 5′ from the standard time meridians, the sun is *never* on the meridian at civil noon. Places less than 4° 5′ away may have one, two, three, or four occasions per year when the sun is practically "on time", but never more than four.

sources of error, and if this were done, the most valuable by-product would be a more accurate determination of the length of the year.

By-Products

(1) This study clearly shows that a well-regulated clock is a steadier time keeper than the sun. Sun days vary in length from 23 hr. 59 min. 39 sec. to 24 hr. 00 min. 30 sec. Civil days are set arbitrarily equal to one another at 24 hr. 00 min. 00 sec., and the mean solar second is 1/86400 of this lapse of time.

(2) Careful comparison of the diurnal paths made by the spot at the vernal and autumnal equinoxes shows that the sun is apparently moving northward at the former time, and southward at the latter. This difference is clearly demonstrable *in the course of two hours!* The sun's rate of change of declination at these times of year is very close to 1 minute of arc per hour. Thus, this observation is consistent with the accuracy claimed. Whereas it is common to disregard this change of declination of the sun during any one traverse of the sky, and to consider that it moves on a diurnal circle about the earth's polar axis, this is not strictly true and the small hourly change of declination *is* observable, especially near the equinoxes, even with simple tools and elementary methods like those described.

(3) The rate of rotation of the earth is directly observable, but care must be exercised in the computation. If the angular speed in degrees per hour is measured from the angular speed of the beam of light, that is, from the rate of motion of the spot divided by the optical lever arm, the result is likely to be too low. For example, as the sun passes the meridian on December 21, the spot has its greatest speed eastward, measured to be 12.5 cm. in 10 minutes for an optical lever 312 cm long. This leads to an angular speed of only 13.8 degrees per hour. By reason of the declination of the sun, this observed angular speed is less than the speed of turning of the earth on its axis. If 13.8° is multiplied by the secant of the angle of declination, 23° 27′, the expected speed of 15° per hour is forthcoming. Thus, one meets interesting problems in solid geometry, such as this one, and is abruptly reminded how dihedral angles must be measured.

(4) An interesting problem presents itself in finding those times of year when the spot moves northward (or southward) by the greatest amount in successive passages of the sun across the meridian. The sun changes declination most rapidly at the equinoxes, but because of the obliquity of the shaft of light striking the floor, the greatest change in the spot (for latitude 40°) occurs not then but about 42 days after the autumnal equinox, or 42 days before the vernal. Presented as a problem in differential calculus, this might be stated as follows: Find the daily rate of change of length of the noon-day shadow of a flagpole of height H at latitude λ, and find at what dates of the year the greatest change occurs in successive passages of the sun across the meridian. When applied to the Empire State Building,

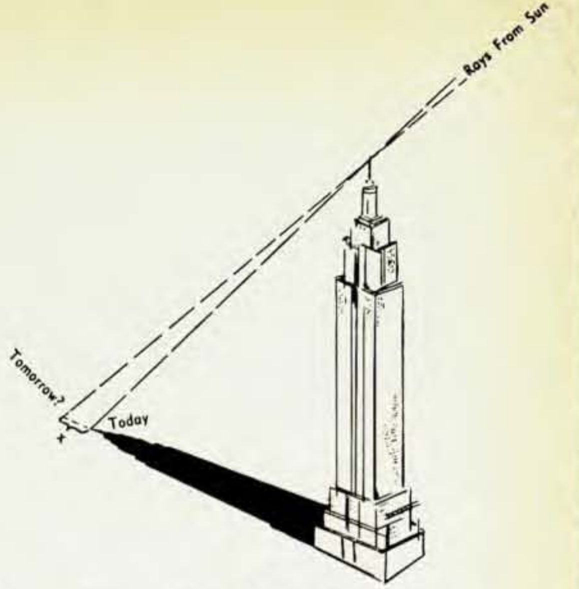

Fig. 2. When does shadow of Empire State Building gain the most between successive noons?

1250 feet high, this leads to a change as great as 11 feet per day. The shadow becomes longer by that amount about November 11 and shortens by that amount about February 7. (Fig. 2.)

(5) Since the bright spot is a pinhole image of the sun, clouds seen crossing the sun appear in this image to go in opposite sense. During one observation, a sharp pinhole image of a passing airplane flashed across the floor as the plane crossed directly in the line from sun to hole. Further, the angular size of the sun's disc can be readily determined to a fair degree of accuracy by measuring the diameter of this pinhole image and dividing it by the radial distance from hole to image.

(6) Now that the analemma has been traced for nearly two years, the need for Leap Year is clearly evident! One might safely say, even from this relatively brief period of observation, that the length of the year is 365.2 ± 0.1 days. This removes some of the mystery of how Julius Caesar's astronomers could have known the length of the year as precisely as they did. It becomes clearly apparent that the invention of leap year can be made *without reference to any clock,* and without reference to a particular theory of the solar system. It is only necessary to observe that the sun does not cross the meridian *just* where it did 365 days before, but that it lags slightly behind by an amount closely approximating one-fourth the angle moved northward or southward in one day. The greater precision of the Gregorian calendar with its omission of three leap years every four centuries has not yet been achieved on this dial in two years! But already the pattern of diurnal lines resembles the interlaced scanning of a television picture, scanned at excessively slow speed.

It was a pleasure to observe the spot on a "Leap Day", February 29, 1956, for the first time, and to rethink the pattern of calendar control by which we manage to make the sun cross the celestial equator on or near March 21, year after year. To my successors, I bequeath the noon observation on February 29, 2056.

(7) The daily path of the spot of light cast on the floor is very nearly part of a hyperbola. This may be

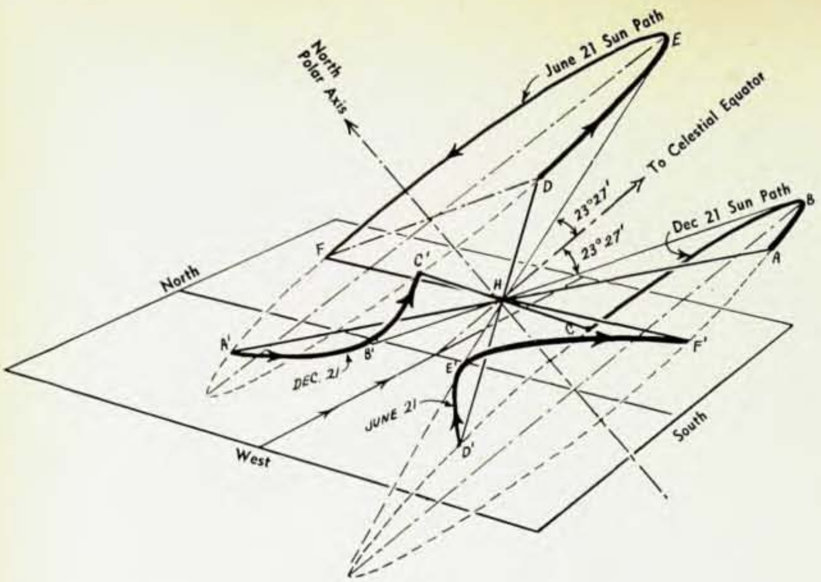

Fig. 3. Conic Section on Floor for Extreme Ray Cones, June 21 and December 21.

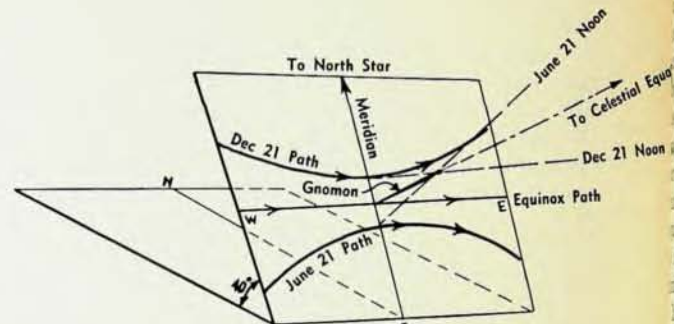

Fig. 4. Extreme Hyperbola for Solstice Shadow of Gnomon.

seen by considering the diagram in Fig. 3, where the sun's diurnal circles *ABC* and *DEF* at the two solstices provide paths on the floor *A' B' C'* and *D' E' F'* which are the intersection of the plane of the floor with the coaxial cones whose common vertex is at *H*, the hole in the window, and whose axis is parallel to the axis of rotation of the earth. As a supplementary experiment to show symmetrically the two branches of the extreme hyperbola that the sun can trace, a drawing board was set up in an open space, inclined at the angle of latitude (40°) and the shadow of a rod or gnomon set perpendicular to the board was traced as the sun crossed the sky during the solstices, June 20 and December 21, 1953. Figure 4 shows the result obtained. This is quite equivalent to tracing the shadow of the tip of a vertical flagpole situated on a level field at the Equator on these two days six months apart.

The Equation of Time

THE equation of time, the curve of correction needed to make sun's apparent time and standard time agree, is an old story to astronomers and navigators. However, it may not be amiss to say a few words for landlubbers about the *physical causes* of the sun's apparently varying motion. Despite knowing better, we still speak of sunrise and sunset, and only by an intellectual effort do we convince ourselves that our own local private coordinate systems are not the most important ones. It is so easy to speak of the sun as rising when we mean that the earth turns to bring it into view. Hence, vagaries in the motion of the earth about the sun are readily translated in our thinking into apparent irregularities in the motion of the sun itself. The sun does not appear to move at a steady rate among the stars, nor to take the same length of time "to get once around the earth" at different times of year. We like to have all civil days the same in length, whereas sun days, noon to noon, differ from one another by as much as 51 seconds. Hence we invent a "mean sun" that keeps a regular rate of passing the meridian day after day, and we express in the equation of time the deviation of the true sun from this mean, or fictitious, sun.

There are two principal causes for departure of the true sun from the mean sun. First, the earth's orbit is elliptical and on this account the speed of the earth in its orbit is not constant. It moves fastest at perihelion, about January 1, and slowest at aphelion, near July 1. In accordance with Kepler's planetary law of areas, "the radius vector connecting sun and planet sweeps out equal areas in equal lengths of time." Newton showed that Kepler's law was a consequence of the principle of conservation of angular momentum for a moving particle constrained by a central force. In a coordinate system "fixed to the fixed stars", the earth because of its own revolution about the sun must always turn approximately one degree more than 360° to bring the sun once more to the meridian. On the average, the excess over 360° is 59 minutes of arc, but near January 1 this becomes 61 minutes; and, in July, 57 minutes. As one minute of arc in the turning of the earth corresponds to 4 seconds of time, this means that the maximum variation in length of day, noon to noon, is about ± 8 seconds from the average *on this account only*. This effect has a periodicity of one full year.

Second, and more important, is the effect of the 23° 27′ inclination of the earth's axis to its orbit. This is the well-known reason for the seasons, as the sun thereby appears to move north and south. But this same geometric fact is also the cause for marked variation in the length of solar days. The more rapidly the sun appears to move eastward among the stars, the longer is the sun day from meridian to meridian. The sun's apparent eastward motion is, however, only that *component* of the earth's orbital motion resolved along the celestial equator. The sun appears to move eastward most rapidly near the solstices when its apparent motion is parallel to the celestial equator; it moves least rapidly at the equinoxes when the sun's apparent motion is inclined as 23° 27′ from the equator. The maximum variation from mean in the length of sun day on this account is about 21 seconds, and the effect has periodicity of one-half year. In Fig. 5 there are graphs of these two effects and a composite of the two curves which shows the variation in seconds, plus or minus, from the mean solar day. The curve of the equation of time, as commonly drawn, is the *integral of this curve,* shown below on a different scale, with arbitrary zero on Christmas Day.

In the light of this discussion, several interesting features of the equation of time curve may be examined. First, days that are just 24.000 hours long are to be found at its peaks and valleys where the *slope* is zero. These occur about February 11, May 15, July 26, and November 6. Second, the longest days are near Christmas time when the *positive* slope of the curve is greatest. In late December and early January, the effects due to inclination of axis and eccentricity of orbit are additive. Here the days are some 30 seconds longer than 24 hours. Near July 1, these effects are subtractive. As the inclination factor outweighs the eccentricity factor in magnitude, the net effect in July is to make days about 14 seconds longer than 24 hours. Third, the shortest days are in September when the *negative* slope of the curve is greatest. Near the equinoxes, the eccentricity factor is virtually zero, but the obliquity factor is maximum, making the days near these two times of year (March and September) about 20 seconds less than 24 hours. At these times of year, the sun appears to progress *least rapidly* eastward among the stars. Finally, where the equation of time curve crosses the axis of ordinates, the sun is "on time" on the standard time meridians at noon, although these days are *not* just 24 hours long.

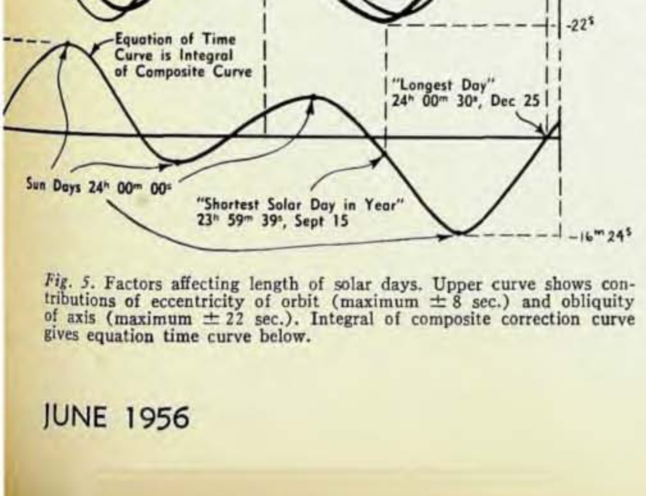

Fig. 5. Factors affecting length of solar days. Upper curve shows contributions of eccentricity of orbit (maximum ± 8 sec.) and obliquity of axis (maximum ± 22 sec.). Integral of composite correction curve gives equation time curve below.

The maximum departure of sun time from civil time occurring about November 3 constitutes the *accumulated gain* from September 1 when the sun is "on time". Throughout this period, sun days are all shorter than 24 hours, with an average deficiency of 15 seconds per day for 65 days. The sun therefore gets ahead of the uniformly running clock (or ahead of the idealized "mean sun") and it crosses each standard meridian on November 3 about 16 minutes and 19 seconds *before* the corresponding standard civil noon. Thereafter, it appears to run at greater than mean speed among the stars until February 10. These hundred days are longer than 24 hours, and the accumulated losses amount to more than 30 minutes, with an average of nearly 20 seconds per day. Although the sun is on the meridian and "on time" again about Christmas Day, its apparent motion among the stars is greatest and sun days (from meridian to meridian) are longest. Other parts of the curve can be examined in the light of the discussion of these portions of it.

Conclusion

IT may be objected that these simple experiments merely plough over ground already cultivated more precisely by modern telescopic means. This is true, but the principal rewards have come through awakening the imagination, pressing for greater accuracy by simple means, and inviting the attention of students and visitors to consider more closely the everyday occurrences of nature. In short, the sun dial's chief function is not to tell time, but to encourage the habit of careful and systematic observation, and deeper understanding of phenomena. Too often, we acquire undue smugness in our intellectual acquaintance with some subject, only to discover that we had not begun to probe its depths, nor to enjoy the interest which direct observation offers.

When Enrico Fermi visited Haverford College in November, 1953, he examined the dial in its early stages of development. The daily progress of the sun southward at that time of year was clearly evident. With the music of hidden mirth and gentle sarcasm that so often characterized his voice, he remarked,

"Ah, I see! And then, if you lose your calendar, you can tell what *day* it is!"

Yes, that and much more!