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A tasty introduction to packing problems

Understanding how particles of all kinds fill space has applications in physics, engineering, materials design, and even machine learning.

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Imagine that you've just cracked open and enjoyed a bunch of pistachios. You're now left with double their number of empty shells. How big of a container do you need to hold the shells without having them overflow or cramming them in?

Although our pistachio problem may seem frivolous, the underlying mechanisms that govern the packing of geometric shapes are remarkably complex. How objects pack together has intrigued scientists for centuries because the problems are often simple to formulate but usually tremendously difficult to resolve. In 1611, for example, Johannes Kepler conjectured that equally sized spheres have the highest average density when they're packed in a face-centered cubic arrangement or a hexagonal close-packing pattern. It took nearly 400 years to formally prove, with computer-assisted methods, that Kepler's seemingly intuitive idea is correct.

Substantial progress has been made on understanding the packing of spheres and ellipsoids. M&M's candies are almost true ellipsoids, and their packing behavior has been successfully predicted with theory. Yet real-world particles rarely have such simple, regular shapes. Most particles—like pistachios and their shells—are irregular, nonconvex, and elongated. The prediction of how such objects fill space remains a largely open problem.

Jamming and rigidity

Packing problems lie at the intersection of geometry, mechanics, and statistical physics. Packings of irregularly shaped particles are essentially amorphous solids and glasses. They lack crystalline order and, at the microscopic level, resemble liquids. The particles become mechanically rigid above a certain density and undergo a jamming transition in which they cease to move. That transition depends on how the particles are compacted. Spherical particles can exhibit different arrangements that lead to jamming. The compaction protocol is even more significant for complex or anisotropic shapes, such as rods, ellipsoids, and other exotic geometries. For those shapes, the jamming transition can occur over a range of densities and ultimately depends on the directions in which the particles nest, align, and obstruct one another.

The shape of constituent particles influences not just how densely they can pack but also how they transmit forces and resist motion once jammed. In turn, the jamming threshold and the emergence of rigid behavior in packed particles con-

strain which packing patterns and macroscopic packing phases are stable and prevalent in natural and engineered materials. The study of jamming transitions sheds light on how materials deform, fracture, and resist flow under external forces.

The jamming of irregularly shaped particles is critical for various phenomena in soft condensed matter and biology, such as densely organized DNA in a cell nucleus, macromolecular crowding in cytoplasm, and spatial arrangements of tissue cells. In addition, soft-matter and biological systems often have further complications. Some systems are sensitive to stickiness, which is characterized as short-range attractions between particles, or to friction, in which energy is dissipated when objects are in contact. Stickiness and friction naturally enrich the phenomena that eventually emerge in systems of packed objects.

Pistachios have many characteristics that make their packing behavior complicated. A simple experiment helps illustrate why.

Packing pistachios

To explore how efficiently pistachios and their empty shells can be packed, we first loosely filled a graduated cylinder with 613 pistachios, with visible gaps between them. Next, we gave the cylinder a good shake and a few rolls to settle the pistachios into a dense arrangement. We repeated the same process with the empty pistachio shells.

We observed that the unshelled pistachios in the loose and dense states have about the same packing density, about 320 pistachios per liter. For the empty shells, however, the difference is significant: After shaking, the packing density is 27% higher. The increase makes sense: Unlike the unshelled pistachios, the nonconvex shells partially nest and interlock, especially when shaken, as shown in the image.

The container for the empty shells, therefore, can be smaller than the bowl for the unshelled pistachios. If you are feeling a bit lazy and toss the shells in loosely, you will need a container roughly three-quarters (73%) the size of the original bowl. But if you pack the shells tightly by shaking the container, it needs to be only a little more than half (57%) the size of the original.

Beyond physical particles

Besides its importance in soft-matter and biological systems, packing behavior is essential in materials science. The inter-



EMPTY PISTACHIO SHELLS are densely packed in this graduated cylinder. Because the shells nest and interlock—which can easily be seen around the 500 mL level—they occupy a volume that's only 57% of the volume taken up by whole pistachios.

nal structure formed by packed particles directly influences the mechanical, thermal, and flow properties of materials.

In ceramic processing, the way powders are compacted before being melted determines the final material's density and strength. In concrete and asphalt production, the packing of aggregates affects durability, porosity, and load-bearing capacity. In additive manufacturing, the compaction and flow of irregular metal or polymer particles help determine the material's structural performance. Each of those examples relies on controlling how disordered, nonspherical particles organize when constrained. Observations and models are often necessary to understand how to control the particles.

Finally, the statistical behavior of disordered packings has inspired connections beyond materials science. In machine learning, the training of neural networks exhibits characteristics that are also present in glassy systems, such as numerous local energy minima, geometric frustration, and nontrivial correlations. Powerful tools that were developed to study packing and jamming have been adapted to better understand AI and neuroscience models. The work of Giorgio Parisi, who was awarded the 2021 Nobel Prize in Physics, in understanding spin glasses helped enable the fruitful transfer of knowledge from statistical physics to computer science (see *PHYSICS TODAY*, December 2021, page 17).

What may appear then to be a simple geometric problem—how many objects fit in a container—turns out to have far-reaching implications across physics, engineering, computational science, and even snacking.

Additional resources

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- P. Charbonneau et al., "Glass and jamming transitions: From exact results to finite-dimensional descriptions," *Annu. Rev. Condens. Matter Phys.* **8**, 265 (2017).
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