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Quantum entanglement: A modern perspective

It's not your grandfather's quantum mechanics. Today, researchers treat entanglement as a physical resource: Quantum information can now be measured, mixed, distilled, concentrated, and diluted.

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"If two separated bodies, each by itself known maximally, enter a situation in which they influence each other, and separate again, then there occurs regularly that which I have [just] called entanglement of our knowledge of the two bodies."

—Erwin Schrödinger (translation by J. D. Trimmer)

Erwin Schrödinger coined the word *entanglement* in 1935 in a three-part paper¹ on the "present situation in quantum mechanics." His article was prompted by Albert Einstein, Boris Podolsky, and Nathan Rosen's now celebrated EPR paper that had raised fundamental questions about quantum mechanics earlier that year.

Einstein and his coauthors had recognized that quantum theory allows very particular correlations to exist between two physically distant parts of a quantum system; those correlations make it possible to predict the result of a measurement on one part of a system by looking at the distant part. On that basis, the EPR paper argued that the distant predicted quantity should have a definite value even *before* being measured if the theory were to claim completeness and respect locality. However, because quantum mechanics disallows such definite values prior to measuring, the EPR authors concluded that, from a classical perspective, quantum theory must be incomplete.

Schrödinger's 1935 perspective comes closer to the mod-

ern view: The wavefunction or state vector gives us all the information that we can have about a quantum system. About entangled quantum states, he wrote, "The whole is in a definite state, the parts taken individually are not,"¹ which we now understand as the essence of pure-state entanglement. In that same 1935 article, Schrödinger also introduced his famous cat as an extreme illustration

of entanglement: A cat physically isolated in a box with a decaying atom and vial of cyanide represents a quantum state having macroscopic degrees of freedom. If the atom were to decay and trigger the release of cyanide, the cat would die. The quantum-mechanical description of the system is a coherent superposition of one state in which the atom is still excited and the cat alive, and another state in which the atom has decayed and the cat is dead:

$$\frac{1}{\sqrt{2}} \left(\left| \text{atom}, \text{cat} \right\rangle + \left| \text{atom}, \text{cat} \right\rangle \right).$$

The isolated cat-trigger-atom-cyanide system as a whole is in a definite entangled state, even though the cat itself exists as a probabilistic mixture of being alive or dead.

For the three decades following the 1935 articles, the debate about entanglement and the "EPR dilemma"—how to make sense of the presumably nonlocal effect one particle's measurement has on another—was philosophical in nature, and for many physicists it was nothing more than that. The 1964 publication² by John Bell (pictured in figure 1) changed that situation dramatically. Bell derived correlation inequalities that can be violated in quantum mechanics but have to be satisfied within every model that is local and complete—so-called local

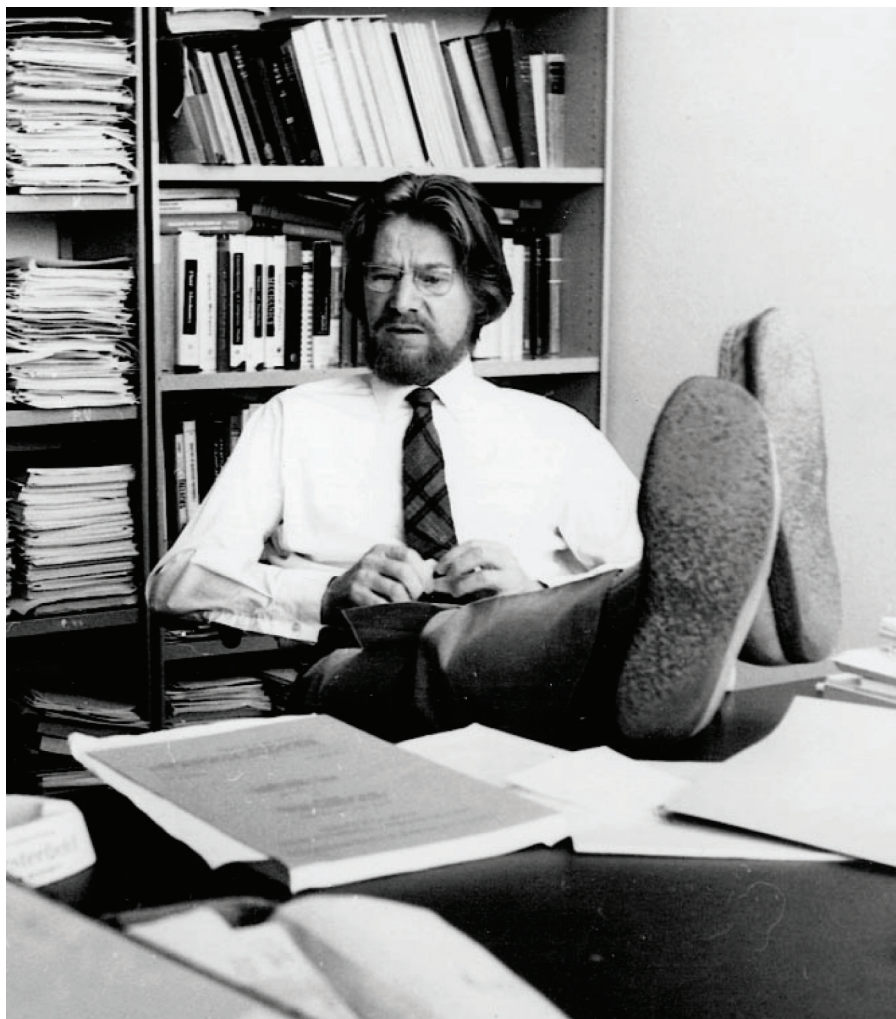


Figure 1. John Bell in repose. His seminal work clarified the difference between correlations generated by entanglement and correlations in local hidden-variable models. Nowadays, quantum information theorists exploit this difference to create advantages that communication protocols using entanglement have over classical ones.

paradoxical feature of quantum mechanics, but as a physical resource for quantum-information processing and computation. A whole zoo of various kinds of pure and mixed entangled states may be prepared—well beyond the simple pure-state superpositions that Schrödinger envisioned.

And those mixed entangled states

hidden-variable models. Bell's work made it possible to test whether local hidden-variable models can account for observed physical phenomena. Early and ongoing recent experiments³ showing violations of such Bell inequalities have invalidated local hidden-variable models and lend support to the quantum-mechanical view of nature. In particular, an observed violation of a Bell inequality demonstrates the presence of entanglement in a quantum system.

In 1995, Peter Shor at AT&T Research discovered that, for certain problems, computation with quantum states instead of classical bits can result in tremendous savings in computation time.⁴ He found a polynomial-time quantum algorithm that solves the problem of finding prime factors of a large integer. To date, no classical polynomial-time algorithm for this problem exists.

Shor's breakthrough generated an avalanche of interest in quantum computation and quantum information theory. In this context, a modern theory of entanglement has begun to emerge: Researchers now treat entanglement not simply as a

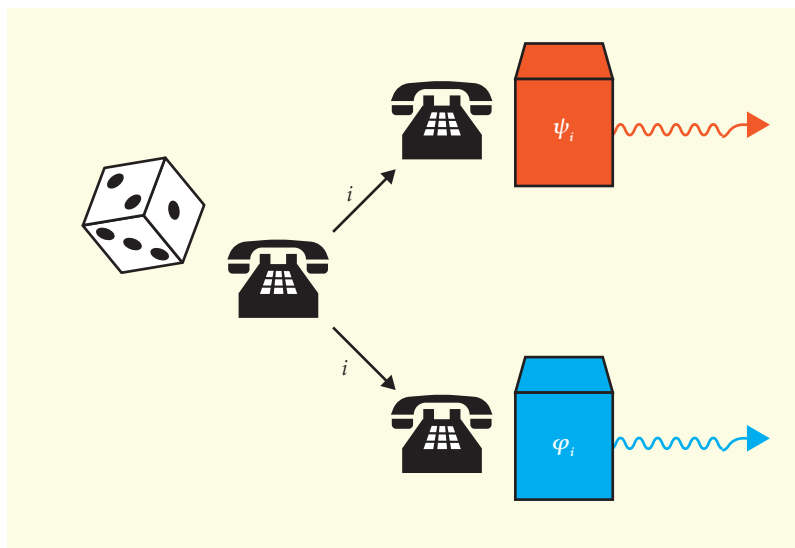
may be measured, distilled, concentrated, diluted, and manipulated. A surprisingly rich picture of entanglement is now taking shape.

Entanglement for the 21st century

The discovery of quantum teleportation by IBM researcher Charles Bennett and five collaborators in 1993 marks the starting point of the modern view. In quantum teleportation (see the article by Charles Bennett in *PHYSICS TODAY*, October 1995, page 24), an experimentalist, Alice, wishes to send an unknown state $|s\rangle = \alpha|0\rangle + \beta|1\rangle$ of a two-level quantum system to another experimentalist, Bob, in a distant laboratory. The two-level system could refer, for example, to the polarization of a single photon, the electronic excitation of an effective two-level atom, or the nuclear magnetic spin of a hydrogen atom. Alice and Bob do not have the means of directly transmitting the quantum system from one place to another (for photons, this could be the case when using a high-loss optical fiber), but let us imagine that they do share an entangled

QUANTUM ENTANGLEMENT

Figure 2. Classically correlated, or separable, quantum states are generated when Alice (red) and Bob (blue) locally prepare quantum states ψ_i and φ_i , depending on the result i of a classical random number generator. If the correlations in a bipartite quantum state *cannot* be produced by such a procedure, then the state is considered entangled.



state. Consider the case in which Alice and Bob each have one spin of a shared singlet state of two spin- $\frac{1}{2}$ particles $|\Psi^-\rangle = 1/\sqrt{2}(|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle)$, also called an EPR pair. Alice can transmit her spin $|s\rangle$ to Bob by performing a certain joint measurement on her spin state $|s\rangle$ and her half of the EPR pair. She tells Bob the result of her measurement and, depending on her information, Bob rotates his half of the EPR pair to obtain the state $|s\rangle$. The teleportation protocol demonstrates that the resources of classical communication and the sharing of prior EPR entanglement are sufficient to transmit an unknown spin state $|s\rangle$. (For the experimental realization, see *PHYSICS TODAY*, February 1998, page 18.)

The spin-singlet EPR state that Alice and Bob share in quantum teleportation is called a maximally entangled state. Even though the two spins together constitute a definite pure state, each spin state is maximally undetermined or mixed when considered separately. In mathematical terms, Alice's local density matrix—obtained by tracing over Bob's spin degrees of freedom, $\text{Tr}_B(|\Psi^-\rangle\langle\Psi^-|)$ —has equal probability for spin up and spin down. In keeping with Schrödinger's understanding of entanglement, one measures the amount of entanglement in a general pure state φ in terms of the lack of information about its local parts. The von Neumann entropy $S(\rho) = -\text{Tr}(\rho \log \rho)$ is used as a measure of that information. In other words, the entropy of entanglement E of the pure state φ is equal to the von Neumann entropy of, say, Alice's density matrix $\rho = \text{Tr}_B|\varphi\rangle\langle\varphi|$.

Mixed entanglement

In the quantum teleportation scenario, we imagined, unrealistically, that Alice and Bob shared an EPR pair free of noise or decoherence. More generally, Alice and Bob have quantum systems that interact directly or through another mediating quantum system—like Rydberg atoms in a laser cavity that interact via photons, or two ions in an ion trap

that interact through phonon modes of the trap.⁵ A related example of interest in quantum computation is an array of interconnected ion traps, each holding a small number of ions that are coupled by traveling photons or by ions that are moved between the traps.⁶ The interaction, or “quantum link,” between a pair of systems is subject to noise or decoherence through photon loss or heating of the phonons, for instance. For simplicity, assume that Alice and Bob's local operations on the quantum systems—operations on the ions in a single trap, say—are perfect, and their exchange of classical information is also perfectly noise free. That idealization enables one to measure the strength of the quantum link between the systems.

An essential question is, Given unavoidable noise levels, is it possible to establish a strong quantum link—a set of pure EPR pairs, in other words—between two systems? If it is, then the noise is weak enough to permit the error-free exchange of quantum information between the systems, since the teleportation through the generated EPR pairs will be error free. That capability may come at a certain cost, determined by the amount of noisy interaction required to generate an EPR pair. If it is not possible to generate EPR pairs, that decoherence in the system imposes a fundamental limitation on our ability to perform quantum information processing.

The possibility of generating shared EPR entanglement in noisy environments is not only of interest in entanglement theory, but is crucial for the realization of long-distance quantum communication⁷ and possibly large-scale quantum computation. For example, it was recently shown⁸ that fault-tolerant quantum computation can be achieved in the presence of very high noise levels in the interaction link—a link can have an error rate of two-thirds—between quantum systems that are “small” in a particular sense, if one assumes that local quantum processing on each end is (almost) error free.

Pure quantum states have their entanglement quantified

fairly intuitively by considering the degree of local “mixedness” or entropy. However, mixtures of entangled and unentangled states are murkier: Recognizing which mixtures are still entangled may be difficult. So, just what physical systems can we call “entangled”? An operational description—expressing entanglement in terms of its negation—is helpful. Suppose that Alice and Bob, working in their distant labs, each receive the same random number over the phone. Depending on the random number, each of them locally prepares a certain quantum state. The physical state of their whole system, expressed as a density matrix, typically exhibits correlations between the two systems. However, those correlations would be classical, since they arise from classical random numbers. A quantum state that can be prepared in this way *over the phone* is called “unentangled” or separable, and such a state can be mathematically expressed as a mixture of unentangled pure states (see figure 2). Conversely, a state is “entangled” if it cannot be prepared over the phone, but requires coherent interaction between the two systems or the transmission of superpositions of quantum states.

Measures of noisy entanglement

For mixed states, it is harder to establish a good measure of entanglement, since such a measure has to distinguish between entropy arising from classical correlations in the state—a state of thermal equilibrium, for example—and local entropy due to purely quantum correlations. Two measures of entanglement that have explicit physical meaning in the processing of quantum information have emerged from the quantum-link notion just described: the entanglement cost $E(\rho)$ of a quantum state and the distillable entanglement $D(\rho)$ of a quantum state, first defined in reference 9.

Assume that Alice and Bob have created, using their noisy link, many (n) shared copies of an entangled quantum state ρ ; we denote such a collection as $\rho^{\otimes n}$. To distill some EPR pairs from those copies, Alice and Bob perform several rounds of local, error-free operations to their parts of the copies and communicate their measurements (or other classical data) to each other. Such a protocol is called entanglement distillation; figure 3 illustrates one round of such a scheme. The aim is to produce fewer states that are, however, more entangled than the initial ones. Ideally, the protocol produces nearly perfect maximally entangled EPR pairs in the limit of a large number of input states $\rho^{\otimes n}$ with $n \rightarrow \infty$. The distillable entanglement $D(\rho)$ is then the number

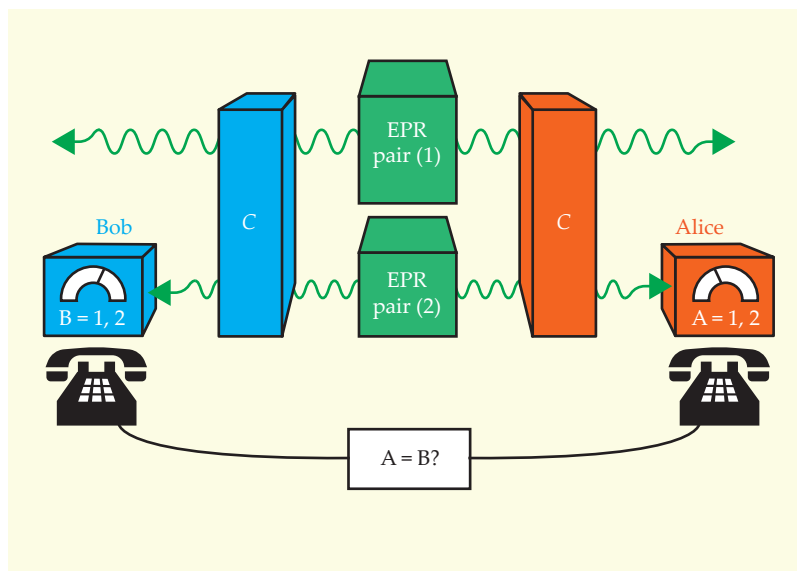


Figure 3. Entanglement distillation—the conversion of many noisy less-entangled states into fewer, more-entangled ones. Imagine two Einstein-Podolsky-Rosen pairs that pick up noise when their parts are transmitted to Alice and Bob. Assume that the noisy states are still entangled. Alice and Bob can use the following protocol to increase the entanglement: (i) each of them applies a controlled-shift operation C to the states sent to them; the shift operation acts on the upper green system (1) and the lower green system (2). For i and $j = 0, 1$, $C|i\rangle_1 \otimes |j\rangle_2 = |i\rangle_1 \otimes |i \oplus j\rangle_2$, where $\oplus$ means addition modulo 2. (ii) Each measures the lower EPR(2) pair in the $\{|0\rangle, |1\rangle\}$ basis and they compare their results. If the outcomes are the same (checked over the phone), the entanglement in the first EPR pair will have increased. The various ways of iterating the procedure to distill more entangled states are known as recurrence protocols⁹ or entanglement pumping.⁸

of such EPR pairs that can be extracted per copy of ρ in this asymptotic limit.

The reverse process also has physical meaning. What is the smallest number k of EPR pairs that Alice and Bob initially need to create a set of n copies of ρ for $n \rightarrow \infty$ by local error-free operations? This asymptotic ratio k/n is the second measure of entanglement, the entanglement cost $E(\rho)$.

Reversible and irreversible manipulation

Attentive readers may have noticed a quirk in our notation: The formalism uses the same symbol E to denote both the entanglement cost for general states and the entropy of entanglement for pure states. The notation coincidence is harmless since the creation cost of a pure state equals the local entropy of entanglement E . Furthermore, for a pure state φ , it turns out that $E(\varphi) = D(\varphi)$ (see box 1 on page 44). Physically, this means that the process of entanglement dilution—converting EPR pairs into lesser entangled pure states φ —can be reversed without loss of entanglement. The reverse process is called entanglement concentration and it produces $D(\varphi)n = E(\varphi)n$ EPR pairs from an initial supply of n states φ .

For mixed states, D is believed to be generically less than E , which implies that the preparation of mixed states

Box 1. The law of large numbers and interconvertible entanglement

Suppose one generates a bit string of length k by k realizations of a binary random variable that takes the value 1 with probability p and the value 0 with probability $1 - p$. By the law of large numbers, among the k -bit strings there exist typical strings that have a high probability of occurring—ones in which approximately $pk + O(\sqrt{k})$ bits are 1 and $(1 - p)k$ bits are 0, for instance—and atypical strings, the string of all zeros, for example. The key to understanding the protocols of pure state entanglement concentration and dilution¹⁸ is this typicality of sequences.

Suppose Alice and Bob would like to convert some shared entangled states $\varphi^{\otimes k}$ with $|\varphi\rangle = \sqrt{p}|11\rangle + \sqrt{1-p}|00\rangle$ to a smaller supply of Einstein-Podolsky-Rosen (EPR) pairs Ψ^- . In other words, suppose they wish to *concentrate* their entanglement in fewer qubits. Alice and Bob will each do a local measurement that counts the number of ones in a bit string (but not which bits are ones). With high probability—approaching 1 as $k \rightarrow \infty$ —they both have pk as their measurement outcome, indicating that pk bits out of k are one. With that outcome, Alice and Bob will have obtained a quantum state whose local density matrix has eigenvalues that are all equal which number approximately

$$\binom{k}{pk} \approx 2^{kH(p) - O(\sqrt{k})} = 2^{kH(\varphi) - O(\sqrt{k})}.$$

Here, $H(p)$ is the Shannon entropy of the distribution $(p, 1 - p)$. Thus Alice and Bob can make a local change of basis (a unitary rotation) and truncate the dimension of the space to 2^n and obtain $n \approx kE(\varphi) - O(\sqrt{k})$ EPR pairs.

In the reverse process of dilution, one converts n EPR pairs into k states φ by quantum teleporting an approximation φ_k to $\varphi^{\otimes k}$ from Alice to Bob using the EPR pairs. In the local spectrum of the state $\varphi^{\otimes k}$, there exist typical eigenstates, with approximately pk bits equal to 1 and $(1 - p)k$ bits equal to 0, and atypical eigenstates. The approximation φ_k is obtained from $\varphi^{\otimes k}$ by truncating the local spectrum to the eigenstates that are in this *typical* subspace. The dimension of this typical subspace is $2^{kH(p) + O(\sqrt{k})}$ and therefore the state φ_k can be teleported using $n \approx kE(\varphi) + O(\sqrt{k})$ EPR pairs. In the limit of large k , the conversion ratios k/n of the dilution and concentration protocols will be the same and thus prove the asymptotic reversibility of the processes.

from EPR pairs is a process involving an irreversible loss of entanglement. Curiously, the $D < E$ conjecture has only been proven for some special classes of mixed states.¹⁰

In 1998, the Horodecki family of Gdańsk, Poland (father Ryszard and sons Paweł and Michał), identified a class of entangled states that exhibit an extreme form of irreversibility. They proved that no entanglement can be distilled ($D = 0$) from these “bound entangled states.”¹¹ And for a large set of states from that class, irreversibility was established by proving that entanglement is required to prepare the states $E > 0$.

Consider the metaphor illustrated in figure 4. If EPR pairs were nodes connected by lines or strands that represent quantum correlations between particles, then one could think of mixed entanglement as entanglement in which the strands are simply mixed up. The mixing may make it hard to reconstruct which particle of Alice is entangled with which particle of Bob. Cutting a few strands reduces the clutter, but every line cut represents an EPR pair lost (compare this process with the distillation protocol in figure 3). Bound entangled states are those mixtures that are so thoroughly mixed up that every single line has to be cut to remove the noise or clutter from the system. But, when every line is cut, no entanglement remains to be distilled.

“Black holes” of quantum information

Because the modern theory of entanglement treats quantum states as physical resources for processing information, one might consider them hierarchically. A simple and ideal world would have only two classes of quantum states: unentangled, classically correlated states that are useless as a resource in quantum teleportation and don’t violate any Bell inequalities, and entangled states whose distillation rate D measures their usefulness in quantum teleportation. If the distillation rate D is nonzero, one can distill from such states some EPR pairs, known to violate Bell inequalities.

Bound entanglement tells us that life is not so simple. Bound entangled states are costly ($E > 0$), but useless in various quantum-information-processing protocols like teleportation. Furthermore, there is evidence that bound entangled states do not violate any Bell inequalities.

In those two senses, bound entangled states are the “black holes” of quantum information theory. Entanglement goes in but is impossible to recover. And like black holes in the theory of gravitation, bound entangled states test the limits of our understanding and puzzle us by their intrinsic irreversibility.

Bound entanglement and partial transposition

In what sense are bound states so thoroughly mixed up that no entanglement at all can be extracted? Bound entangled states behave intrinsically differently from every other entangled state: They remain physical under the *unphysical* operation of partial transposition.

Researchers realized that they could characterize entanglement in terms of how states behave under certain unphysical operations.¹² In 1996, Asher Peres at the Technion–Israel Institute of Technology in Haifa, Israel, noted that matrix transposition is just such an unphysical operation when applied to entangled states. Taking the transpose of a system’s density matrix produces another density matrix—a physically valid result. And taking the transpose of, say, Bob’s part of an unentangled state $\psi_A \otimes \psi_B$ yields another physically valid quantum state, since each part of the quantum state can transform separately; ψ_A is not changed, and the density matrix of ψ_B is transposed. But when ap-

plied to part of a pure entangled state, matrix transposition produces an unphysical result. (For details, see box 2 on page 46.)

Peres conjectured that partial transposition was the defining criterion for entanglement. In other words, all entangled states—pure or mixed—should map onto unphysical states by partial matrix transposition, and all unentangled states will remain physical under the same operation.

Remarkably, the truth of that conjecture depends on the dimension of the underlying Hilbert spaces or phase spaces. If one considers the state of two spin- $\frac{1}{2}$ particles, the polarization degrees of freedom of two laser beams, or two modes of a light field having a Gaussian Wigner function, then, indeed, all entangled states map onto unphysical states by partial transposition. However, for two spin-one (or higher-dimensional system) particles or a Gaussian light field with at least two modes for both Alice and Bob, that is no longer true in general; there exist entangled mixed states that pass the “partial transpose” test and have therefore lost an essential property of entanglement.

The loss of that property is precisely what the Horodecki family showed would lead to a zero distillation rate D . Entangled states that pass the partial transpose test are the bound entangled states in which the entanglement is forever locked or “bound” inside.

Entanglement witnesses

Given that entanglement can be such a subtle property of quantum states, just how can one distinguish between entangled and unentangled states? A violation of a Bell inequality has been the traditional telltale sign of entanglement in a quantum system. Examples of such experiments³ used pairs of entangled photons created from nonlinear optical processes, especially parametric down-conversion; the polarization degrees of freedom of the emitted photons carried entanglement. Alice and Bob checked for a Bell inequality violation by using local analyzers to measure the polarization of the photons along various angles.

Unfortunately, many quantum states, including the set of bound entangled states, are not known to violate any Bell inequality. And considering the existing limitations on experimental control of quantum systems, experimentalists prefer to check for entanglement using the fewest possible local measurements. The theoretical framework of an entanglement witness, of which a Bell inequality is a particular example,¹³ addresses those two issues. The defining property of an entanglement witness W is that its expectation value with respect to any unentangled state ρ is always nonnegative, $\text{Tr}(W\rho) \geq 0$. At the same time, there exist entangled states σ for which $\text{Tr}(W\sigma) < 0$. Measuring W on a quantum state σ and finding a negative expectation value thus establishes the entanglement of σ . The good news is that there is an entanglement witness for every entangled state; given an experimental means, any entanglement, bound or otherwise, can be detected. The bad news is that entanglement witnesses are

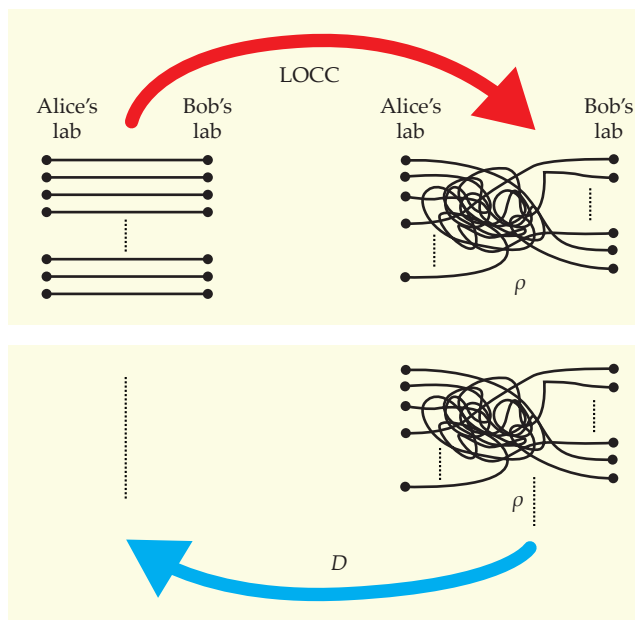


Figure 4. Irreversibility in noisy entanglement. An entangled EPR pair is represented by a single line or strand connecting two nodes or particles, one each in Alice and Bob’s labs. The red arrow signifies the creation of some mixed entanglement from the single strands by local operations on the particles (and classical communication, on the phone, say); the process is abbreviated LOCC. One state ρ that has five particles for both Alice and Bob is created. The entanglement cost is the number of EPR pairs that is needed per single noisy state ρ , in this case $\frac{7}{5}$ because Alice and Bob began with seven EPR pairs. But how does one reverse the process and extract some single strands—EPR pairs—from the noisy mixtures? The distillation rate D is the number of EPR pairs that can be extracted per noisy state ρ . *Bound entangled* mixtures are those that are so thoroughly mixed up that there are no means to extract any single strands. In other words, for a bound entangled state the blue arrow representing the distillation rate D is zero.

nonlocal observables. Nevertheless, one can measure the expectation value of W by measuring the expectation value of a number of local observables W_i , such that $W = \sum W_i$. Research is under way to determine the minimal number of local measurements for a given witness.¹⁴

Bell’s communication advantages

Given the framework of entanglement witnesses, what is special about Bell inequalities? Although they can be considered a type of entanglement witness, Bell inequalities do not, strictly speaking, test for entanglement but for a departure from local hidden-variable theories. Interpreted as such, Bell inequalities have taken on a whole new life in quantum-communication science. Researchers consider remote parties who have to carry out a certain task with minimal communication between them. One compares the amount of communication necessary if those parties are given shared random bits (that can be viewed as local hidden variables) or an

Box 2. Partial matrix transposition and time reversal

Matrix transposition on density matrices is closely related to the operation of time reversal—represented by an anti-unitary operation—in quantum mechanics. The time-reversal operation reverses the momenta, including angular momenta and spin, of a quantum system. It is possible to represent the operation by complex conjugation that maps the momentum operator $\hat{p} = -i\hbar/dx$ onto $\hat{p} = i\hbar/dx$. Applied to Hermitian density matrices, complex conjugation is identical to matrix transposition $T: \rho \rightarrow \rho^T$ in a given basis. When applying this operation on an entire density matrix ρ , one obtains another valid density matrix $\rho^T = \rho^*$ with nonnegative eigenvalues. But when the transposition operation is applied “partially” to half of a joint system—the maximally entangled state $|\Phi\rangle_{AB} = 1/\sqrt{2}(|00\rangle + |11\rangle)$, for example—then one may no longer end up with a valid quantum state. Indeed, transposition in the $\{|0\rangle, |1\rangle\}$ basis on Bob’s half of the state Φ_{AB} (and the identity operation I_A on Alice’s half) gives $(I_A \otimes T)(|\Phi\rangle\langle\Phi|) =$

$$(I_A \otimes T) \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

a matrix that has a negative eigenvalue, and is therefore unphysical. The relevance of partial transposition for detecting entanglement in a quantum state was first noted by Asher Peres in 1996. He observed that any unentangled state remains unentangled under partial transposition, because a product state $|\varphi_A\rangle \otimes |\varphi_B\rangle$ is mapped onto another product state $|\varphi_A\rangle \otimes |\varphi_B^*\rangle$ by transposition of Bob’s system.

entangled quantum state. Sharing entangled states leads to savings in communication precisely because the correlations in quantum states cannot always be adequately described by local hidden-variable theories¹⁵ (see the article by Andrew M. Steane and Wim van Dam, in *PHYSICS TODAY*, February 2000, page 35).

What lies beyond

The efforts of the quantum information theorists over the past eight years would come to little if the theory were not supplemented by an ability to create and manipulate entanglement in the lab. There is a rapidly growing list of physical systems—optical and atomic systems especially—in which it is possible to prepare various kinds of entangled states. As discussed previously, the use of photonic degrees of freedom, such as polarization or momentum, has been a long-time favorite way to create entanglement.³ Entangled states consisting of the quadrature observables of different modes of light have been prepared in optical parametric oscillators and optical fibers.¹⁶ Entanglement in the states of motion of the valence electrons⁵ of trapped ions or of

Rydberg atoms in cavity quantum electrodynamics has involved up to four different atoms. Another promising avenue is the recently observed entanglement of large ensembles of atoms.¹⁷

This short review showcases just a few striking facets of the modern theory of entanglement. Most notably, entanglement shared between more than two subsystems is outside our scope here. The broader study of entanglement between many subsystems may lead the field to better understand the role of large-scale entanglement in quantum computation or quantum many-body systems.

We have focused on the role of entanglement in the transmission of quantum information. Entanglement also proves useful, however, when the goal is to transmit classical information as efficiently as possible. Researchers are studying many measures of mixed entanglement beyond the two most prominent measures discussed in this review. As for bound entanglement, there is some evidence that it may have a role to play as “helper” entanglement, useless by itself, but useful when combined with other sources of entanglement. For entanglement-theory overview articles that highlight the field, see volume 1 of *Quantum Information and Computation* (July 2001).

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