

The behavior of thin curved sheets is ironed out

Two simple geometric rules predict the wrinkle patterns of curved surfaces that are flattened.

Cartographers have long contended with the distortions in a map when the curved two-dimensional surface of Earth is projected onto a flat plane. The Mercator projection, for example, makes landmasses and other features far from the equator appear larger than they really are. Equal-area projections, on the other hand, distort the shapes of geographic features.

When curved surfaces are flattened, wrinkles form because of the mismatch in lengths on a curved surface and those on a flat plane. Nonlinear mechanics and tension-field theory do a good job of describing wrinkles made by tension acting on a shell—that is, a curved sheet. The tension partially stabilizes the crests and troughs of a wrinkle. But the approach fails to explain why wrinkle patterns appear when there's no external tension at all, such as when an elastic sheet that's just a few tens of nanometers thick is confined to a spherical substrate.

Ian Tobasco (University of Illinois at Chicago), Joseph Paulsen (Syracuse University), Eleni Katifori (University of Pennsylvania), and their colleagues recently considered the issue from a mathematical angle. The simple set of geometric rules they developed are an exact solution for the wrinkle patterns formed on flattened, curved sheets. Reassuringly, the rules' predictions agree with numerous experiments and simulations.¹

Isosceles triangles

For the last several years, the researchers had been independently studying wrinkle patterns in shells. In 2017, for example, Desislava Todorova and others in Katifori's group found that the wrinkle patterns that form on thin elastic sheets share similar physics to stripe-patterned liquid crystals.² Around the same time, Graham Leggat and Yousra Timounay, working in Paulsen's group, succeeded in manufacturing ultrathin curved sheets in the lab. And Tobasco was beginning to develop a mathematical framework to explain the phenomena.



FIGURE 1. RANDOM WRINKLES formed on this thin sheet with a diameter of 42 mm. It was cut from a curved elastic sphere with a radius of curvature of 77 mm before being placed on a flat liquid surface. Other shapes form ordered wrinkle patterns. Both types of patterns can now be predicted with a simple pair of geometric rules. (Courtesy of Monica Ripp, Paulsen Lab, Syracuse University.)

"But the bigger story started to emerge once we crossed paths at a SIAM [Society for Industrial and Applied Mathematics] meeting in summer 2018," says Paulsen. "We realized that our ideas and results could be combined into something larger." The collaboration considered the wrinkles that would form on a shape that was cut out of a thin curved sheet and then confined to a planar liquid surface. Some regions of the sheets had ordered repeating structures and others, like those in figure 1, were more disordered.

Despite the seeming complexity of the wrinkles, the researchers found that the patterns could be predicted using two rules. For negative-curvature shells—think of a horse saddle, for example—the first rule predicts that wrinkles form along line segments perpendicular to the shell boundary, as shown in blue in figures 2a and 2d. The wrinkles meet

along the medial axis, defined as the set of points that have two or more closest edge points. Notably, those segments make up the equal legs of isosceles triangles (figures 2b and 2e).

The second rule predicts that for globes and other positive-curvature surfaces (figures 2c and 2f), wrinkles form along the opposite legs (yellow) of the isosceles triangles. As a result, the wrinkle patterns in positively and negatively curved sheets are related: The pattern in one shell can be used to deduce that of its oppositely curved twin. "This reciprocal relationship was one of the most surprising observations," says Tobasco.

The isosceles triangles predict the location of the areas with ordered wrinkle patterns, even for nonuniformly curved surfaces, like an egg. The exact amplitude of the wrinkles' crests and troughs depends on the shell's specific curvature

and some other parameters, but the overall layout of the wrinkle pattern depends only on the sign of the curvature.

Disordered wrinkle patterns also follow the rules. The disordered areas are reciprocally related to a point on the medial axis that has three or more closest boundary points, and the area is bounded by the yellow polygon in figures 2e and 2f. Although the statistics of the disordered patterns can't be predicted, the geometric rules do identify their locations on the cutout shape.

Smashed glass

To test the predictions of the new geometry-based rules, Paulsen, Leggat, and Timounay spin-coated polystyrene films onto curved glass surfaces. Then they observed the wrinkles that formed when curved sheets of various shapes were cut from the films and subsequently placed over a flat liquid surface.

"We started working with concave glass lenses, which are well controlled and could be purchased in a variety of curvatures," says Paulsen. "However, it turned out to be very difficult to separate the shells from these surfaces. So we had to develop a protocol for peeling the film off the substrate without tearing it or damaging it in any way."

Another challenge was making a negatively curved shell. "It seems simple at first—you can just spin-coat onto a saddle-shaped substrate. But we could not find a well-controlled glass substrate with uniform curvature," says Paulsen.

Once Tobasco predicted that the patterns depend only on the sign of the curvature, though, more substrate options became available. The spout of a laboratory beaker, for example, has negative curvature. Paulsen recounts that "Yousra put a glass beaker in a plastic bag, smashed it, and carefully selected a shard that could be used to spin-coat a film on. She formed films on this shard, floated them onto water, and the wrinkle patterns matched the theory beautifully!"

Coverage maximization

In a mathematical paper published last year, Tobasco showed that, in a limit where the wrinkles are infinitely fine, the wrinkle patterns could be derived as a consequence of the curved sheet trying to cover as much of the liquid surface as possible.³ That coverage maximization is driven by energy minimization. "There's a trade-off

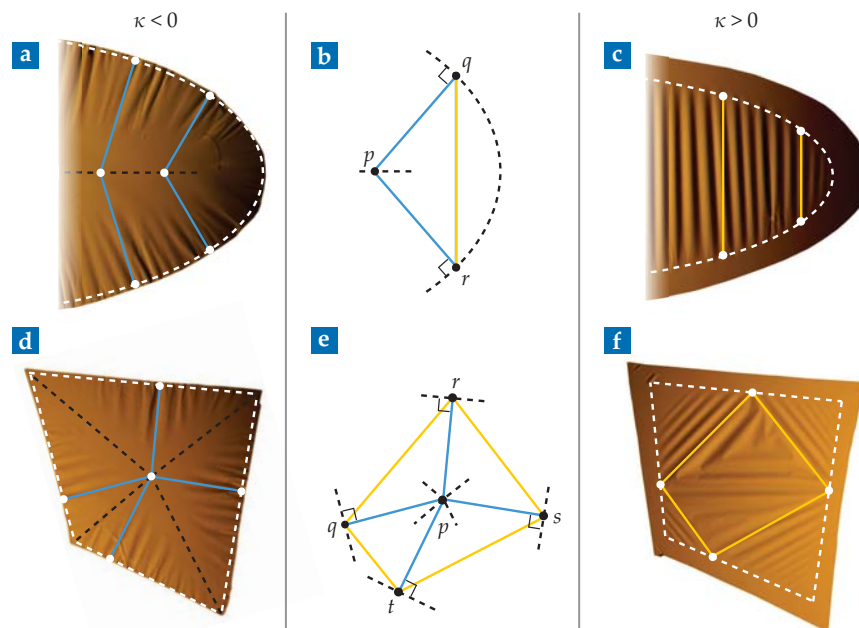


FIGURE 2. THIN CURVED SHEETS that are pressed onto a flat surface often exhibit ordered wrinkle patterns. **(a, d)** For sheets with negative curvature κ , wrinkles form along line segments (solid blue lines) that follow the shortest path to the sheet boundary (dashed black line) from a medial axis (dashed white line)—the set of points that have two or more closest edge points. **(b, e)** The line segments form the two equal-length sides of isosceles triangles. **(c, f)** Sheets with positive curvature produce wrinkles that form along the triangles' opposite legs (solid yellow lines). The area bounded by the yellow polygon in panel f is characterized by disordered wrinkle patterns. (Adapted from ref. 1.)

between the amount of area you can cover by unfurling the shell and the amount of energy you have to spend by wrinkling," says Tobasco.

Physically, surface tension—the dominant role of the liquid in the experiment—acts to pull the sheet's edges as far apart as possible.⁴ But gravity is also at play on the system, and its effect doesn't obviously lead to coverage maximization. "Before deriving the theory, I had no intuitive guess for how gravity would select the patterns," says Tobasco.

Katifori's group spearheaded the simulations. The team used a finite-element method to study how gravity may affect wrinkle patterns. Katifori and her colleagues found that gravity-driven systems were no different than ones driven by surface tension. The simulations with zero surface tension produced the same coverage maximization and the same wrinkle patterns as in the experiments.

The wrinkle patterns are similar to so-called locking materials. In fact, that similarity was what led Tobasco to the two rules for predicting the patterns. If one pulls at the end of a fitted bedsheet, for example, it initially stretches with only a negligible applied force. Eventually, however, locking materials experience an

abrupt limit where they cannot stretch further unless there's a substantial increase in force.

Wrinkle patterns show more subtle locking behavior. If one pulls a wrinkled sheet perpendicular to the crests and troughs, then the wrinkles disappear. But pull the sheet along the crests and troughs, and the wrinkle pattern locks into place.

Although the new rules make predictions for wrinkles, they may be useful for understanding folds and other microstructures in bulk materials and thicker films. "Being ultrathin is not actually absolutely necessary," says Katifori. "People are working in more intermediate-thickness regimes, and you still see similar patterns. It seems that some aspects of it are true in a very wide range of regimes."

Alex Lopatka

References

1. I. Tobasco et al., *Nat. Phys.* **18**, 1099 (2022).
2. H. Aharoni et al., *Nat. Commun.* **8**, 15809 (2017).
3. I. Tobasco, *Arch. Ration. Mech. Anal.* **239**, 1211 (2021).
4. J. D. Paulsen et al., *Nat. Mater.* **14**, 1206 (2015).