QUICK STUDY

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Trigonometry for the heavens

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The stars and planets move not on a flat surface but on the celestial sphere. So ordinary plane trigonometry isn't adequate to describe their motion.

Hipparchus had a problem. The second-century-BC astronomer wanted to quantify his models for the motions of the heavenly bodies—especially the Sun and Moon—so that he could accurately predict their positions and hence eclipses. To be able to say that the Sun would be in a certain place at a certain time, he needed a trigonometry for bodies that move not in planes but on spherical surfaces.

Scholars do not agree who invented spherical trigonometry. It might have been Hipparchus himself, who is credited with inventing planar trigonometry. Possibly it was Menelaus two centuries later, or even Ptolemy a half century after that. In any case, Ptolemy's astronomical classic *Syntaxis mathematica* contains a fully realized spherical trigonometry, which differed from ours in several ways. For example, its basic function was the chord of a circular arc rather than the sine, which would be invented centuries later in India. Arabic astronomers who deemed the book "majestic" gave the work its better-known name, *Almagest*, long after its AD 140 appearance.

Triangles in curved space

Imagine you and a friend are at the north pole of a sphere. You each head off on a trek to the equator, departing along paths at right angles to each other. When you reach the equator, you both turn toward each other and walk until you meet. As panel a in the figure shows, the two of you have now formed a spherical triangle with three right angles, two on the equator and one at the pole; the angle sum is not 180° as in a plane but 270°. And that sum is not the same for all triangles. For a tiny spherical triangle, which would be nearly flat, the angle sum is just over 180°. For a large one, the angle sum can approach 540°.

Examine the sides of the triangle in panel a and you'll see that they are all 90° arcs of great circles. And you've probably made a crucial mental transition: On a sphere, side lengths are measured in degrees just as angles are. Let's use that understanding to solve an astronomical problem as relevant today as it was to Hipparchus more than two millennia ago. Panel b shows the geometry and defines the terms we'll use. Bold white arcs are on the surface of the celestial sphere, taken to have unit radius and center A; thin colored lines lie within the sphere. The Sun, F, travels along the ecliptic, which intersects the celestial equator (the same plane as Earth's equator) at an angle $\varepsilon = 23.4°$, the same angle that describes the tilt of Earth's axis with respect to its orbital plane. The arcs GH and GC are both 90°, so CH also equals ε. The Sun's longitude, $\lambda = \mathrm{GF}$, determined by the time of year, should give us enough information to calculate the Sun's altitude above the equator, the declination $\delta = \mathrm{FJ}$. But how? Here is a sketch of the proof; additional details are provided in the online version of this Quick Study.

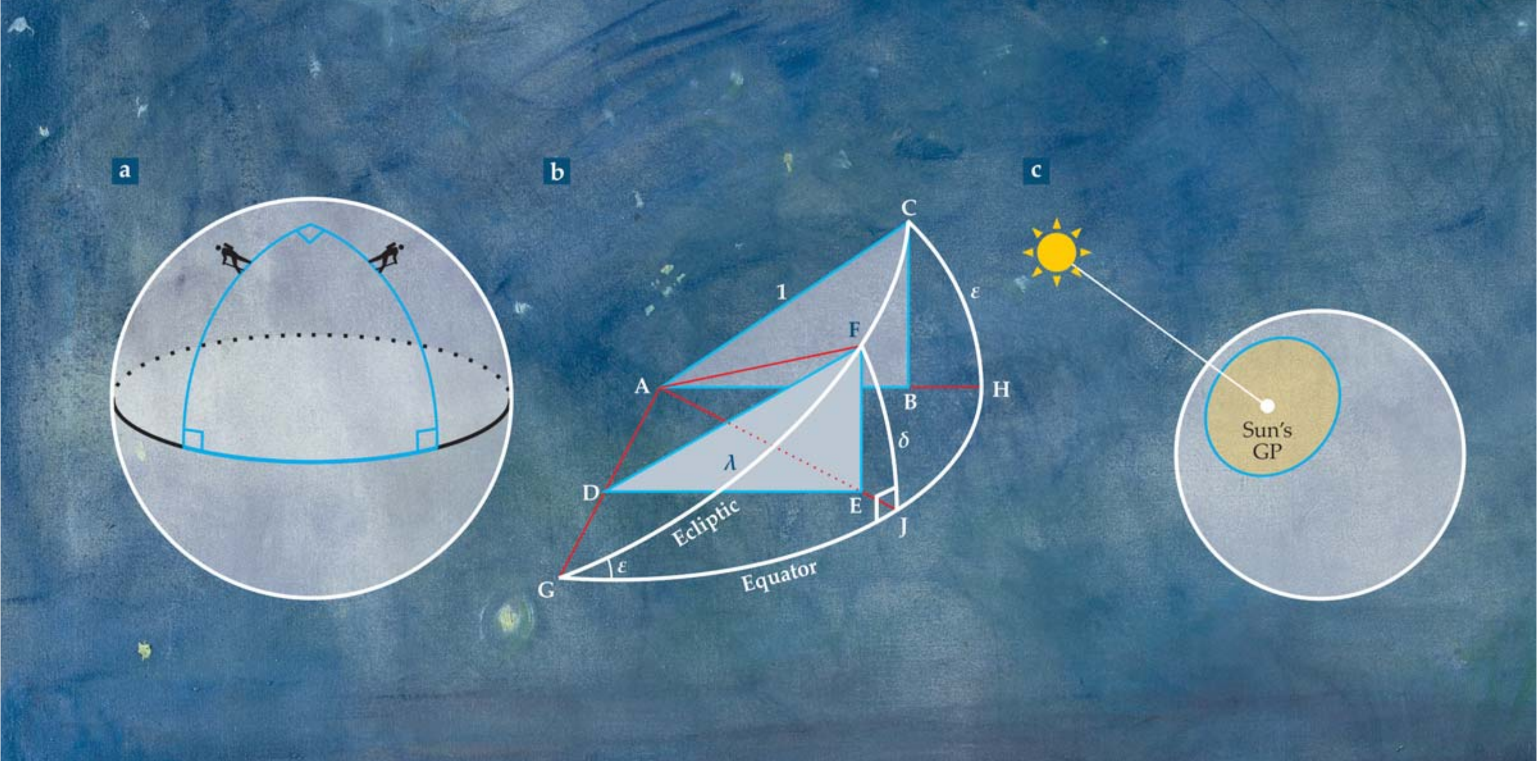
Considering slice HAC, we see that the angle made by the blue lines at A is equal to arc CH, which is ε. Since $\mathrm{AC} = 1$, $\mathrm{BC} = \sin\varepsilon$. Likewise, looking at slice GAF yields $\mathrm{DF} = \sin\lambda$, and slice JAF similarly yields $\mathrm{EF} = \sin\delta$. But triangles ABC and DEF are similar. Thus, $\sin\delta/\sin\lambda = \sin\varepsilon/1$, which is just another way to express the standard declination formula $\sin\delta = \sin\lambda \cdot \sin\varepsilon$.

Nowadays any pocket calculator can handle the declination formula with ease, but before 20th-century technology, astronomers worked well into the day doing declination calculations by hand. In 1614 John Napier announced a solution to the difficulty. With the logarithms he introduced in his *Description of the Miraculous Table of Logarithms*, the laborious multiplication of sines in the declination formula can be expressed in terms of a more manageable sum. Additional information about Napier's book is included online.

From the heavens to Earth

Trigonometry, both plane and spherical, was intended for astronomers; 15th-century astronomer Regiomontanus called it "the foot of the ladder to the stars." But spherical trigonometry has also influenced actions of those with Earth-bound concerns. The earliest occasions were provided by medieval Islamic scholars, who often needed astronomy to resolve demands required by ritual. Examples included predicting the beginning of the sacred month of Ramadan, which is defined by the emergence of the lunar crescent from the glare of the Sun at the time of the new moon, and determining the times of the five daily prayers, some of which required knowledge of the Sun's altitude. Moreover, to pray, a worshipper needed to face toward Mecca, a requirement that was also a problem in spherical trigonometry: Determine the position in the sky of the point in the celestial sphere directly above Mecca. Drop that point to the horizon and face in that direction.

One of the most dramatic stories in the history of mathematics is the 1837 adventure of Thomas Hubbard Sumner. He set off from South Carolina, and three weeks later he needed to sail through St George's Channel between Wales and Ireland. However, miserable weather and obscured skies made him unsure of his position, and potentially fatal rocks awaited along



TRIGONOMETRY ON A SPHERE. (a) For a triangle drawn on the surface of a sphere, the sum of the interior angles can range from 180° to 540°. In this example, the sum is 270°. **(b)** Given the time of year, how high is the Sun above the celestial equator (which is in the same plane as Earth's equator)? The text answers the question, with the help of this construction. **(c)** At any time, the Sun is directly above some point on Earth called the geographic position (GP). By measuring the angle of the Sun with respect to the horizon, a ship's navigator can determine a circle (blue) on which the vessel is located. The text describes how Thomas Hubbard Sumner sailed out of trouble by using that observation. (Background: Detail from *Starry Night*, Edvard Munch, 1893.)

the shores. The clouds parted momentarily, which gave him just enough of a window to measure the Sun's altitude, 12° 10′ above the horizon. Then, with his creativity perhaps sharpened by the stakes of survival, he reasoned as follows: The collection of places on Earth's surface where the Sun is at a given altitude forms a circle whose center is the Sun's geographic position—that is, the point on the surface directly below the Sun at a given time (see panel c). Sumner knew he had to be somewhere on that circle, whose location he could calculate.

Such a circle is called a small circle, not because it is small but because it isn't a great circle. In Sumner's vicinity off the south coast of Ireland, his circle was actually extremely large, very nearly a straight line; navigators call it a line of position. By great fortune, Sumner's line of position passed through the sea in a northeasterly direction and very nearly contacted Smalls Lighthouse off the coast of Wales in a well-charted region. Although Sumner didn't know where on the line he was, all he had to do was to keep traveling along it. He would be assured of eventually spotting Smalls Lighthouse, and from there he could navigate to safety.

A variation of Sumner's ingenious reasoning allows you to pinpoint your location on Earth, no matter where you are. If measuring the altitude of a star determines your location on a small circle on Earth's surface, measuring the altitude of two celestial bodies places you at the intersection of a pair of circles. The two circles intersect at two points, one of which is your location. Almost always, those two points are far away from each other, and if you cannot tell whether you're off the south coast of Ireland or the south coast of India, you have bigger problems than navigation can solve. Nautical almanacs tabulate the positions of several dozen reference stars, the Sun, and several planets, so you have plenty of celestial bodies to choose from. In practice, you can determine a ship's location to well within a mile and can increase the reliability of the method by observing the altitudes of more than two bodies.

Modern technologies such as GPS have rendered traditional practices of spherical trigonometry obsolete other than for hobbyists. But they are making at least a small comeback. At the US Naval Academy in Annapolis, Maryland, one of the last institutions to give up teaching spherical trigonometry back in the 1960s, officers in training are now being instructed in celestial navigation. The potential for GPS systems to be jammed by enemies at times of conflict may cause threatened sailors to turn their eyes to the heavens, not to cry for help from divine powers, but to apply ancient ingenuity to save themselves and their shipmates.

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Additional resources

▶ M. Vanvaerenbergh, P. Ifland, *Line of Position Navigation: Sumner and Saint-Hilaire, the Two Pillars of Modern Celestial Navigation*, Unlimited (2003).

▶ G. Van Brummelen, *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry*, Princeton U. Press (2009).

▶ G. Van Brummelen, *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry*, Princeton U. Press (2013).

▶ M. Blewitt, *Celestial Navigation for Yachtsmen*, 13th ed., rev. by A. Du Pont, Adlard Coles Nautical (2017).

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