COHERENT SYSTEMS

A review of some questions concerning the need for consistency in deriving convenient systems of physical units.

By ***Dudley Williams***

MOST AMERICAN PHYSICISTS have great zest for the study of physical phenomena but complete apathy toward the problems involved in the development of a convenient system of units for general use in describing the phenomena. This attitude is understandable and even defensible, since the phenomena themselves and their interpretation are the subjects of prime importance, whereas the units used in quantitative description are of secondary importance. For example, if some nuclear physicist, for reasons of his own, chooses to publish his results in a table of nuclear properties with masses expressed in long tons, cross sections in acres, and spins in Btu·hrs, these properties are stated definitely and can be expressed without serious difficulty in more conventional units for comparison with the results of other investigators; however, it is admittedly a matter of convenience for all nuclear physicists to employ certain standard units. We all recognize that the establishment of a convenient set of units for general use is desirable but most of us are so preoccupied with our own immediate research interests that we have little time or energy to devote to the matter.

Our colleagues in electrical engineering have devoted considerable attention to the establishment of a convenient system of mechanical and electromagnetic units. The system favored by the engineers is a system first proposed by Giorgi in 1901. This Giorgi or MKS system was recommended by the International Electrotechnical Commission in 1935 and again at its 1950 meeting in Paris, where the ampere was adopted as a fourth unit giving the MKSA system. Meantime, in 1948, the Ninth General Conference of Weights and Measures, on the suggestion of the International Union of Pure and Applied Physics, had also recommended the Giorgi system and adopted the ampere as the fourth fundamental unit.

Further study of the subject is being made by the International Electrotechnical Commission together with representatives of the Commission on Symbols, Units, and Nomenclature (SUN). The results of this study may be considered at the 1954 meeting of the SUN Commission as a whole and may then be submitted to the International Union of Pure and Applied Physics for approval. One of the questions to be considered is whether the MKSA units are to be employed in the *rational* or *nonrational* form.

In order to stimulate discussion of the subject and to elicit expressions of opinion for the guidance of the American representatives, an article giving a brief review of the subject has been suggested. As one who uses MKSA units in an introductory text and CGS units in journal articles, as one who from *Physical Review* tradition uses H in journal articles where he should use B, and as one who on occasion employs the time-honored units *jiggers* and *fifths* for the measurement of volume, the writer was regarded as eminently qualified to prepare such a review since he obviously has no axe to grind and can present an unbiased view of the matter. Here goes!

"Quantities" are the quantitative physical concepts of science. A given physical quantity A such as length, time, energy, electric current, *etc.* is the product of a numerical value or "measure" $\{A\}$ and a unit $[A]$, the unit being a particular reference-sample of the quantity chosen as a standard. Thus,

$$A = \{A\} \cdot [A] \quad (1)$$

Quantity = Numerical Value · Unit

The unit $[A]$ may be chosen quite arbitrarily, but it is desirable to define units in such a way that they are derived from a few fundamental units by equations without numerical factors other than unity and so that the equations between the numerical values of quantities have exactly the same form as the equations between the quantities. For example, the kinetic energy of a body is given by

$$E = \tfrac{1}{2}\, MV^2 \quad (2)$$

where $E = \{E\} \cdot [E]$, $M = \{M\} \cdot [M]$, $V = \{V\} \cdot [V]$, and $\tfrac{1}{2}$ is called a "definitional factor" and is dimensionless. If the units of E, M, and V are defined in such a way that

$$[E] = [M] \cdot [V]^2, \quad (3)$$

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then the equation between the numerical values is

$$\{E\} = \tfrac{1}{2}\{M\}\cdot\{V\}^2.$$

A system of units defined in this way is called a *coherent* system (with respect to the quantities in question). It is constructed by defining the units of a few *fundamental quantities* independently (the *fundamental units*) and defining the units of all other quantities by equations like (3) with no numerical factors other than unity (the *derived units*).

It is a matter of choice as to which quantities are to be considered as fundamental and also how many. It is desirable to have as few fundamental quantities as possible provided no serious inconveniences are encountered; if, however, convenience can be achieved, it is desirable to increase the number of fundamental quantities beyond the minimum. In mechanics it is convenient to have three fundamental quantities and in the "absolute" systems these are *length, mass,* and *time.* The units of several physical quantities in the CGS and in the MKS or Giorgi system are given in Table I. It will be noted that the practical energy and power units occur as part of the coherent Giorgi system.

In electricity and magnetism, it is possible to derive all the electrical and magnetic units from the three fundamental mechanical units. This has been done in several ways. If one writes Coulomb's Law in the form

$$F=\frac{Q_s\cdot Q_s}{r^2} \tag{4}$$

for empty space, a *three-dimensional electrostatic system* of units is established. In this system, the charge unit $[Q_s]$ and current unit $[I_s]$ are expressed in terms of the mechanical units and *dimensions must be attributed to the permeability of empty space.* The electrostatic units are summarized in Table II. If one chooses to start with the equation

$$F=\frac{2\,I_m\cdot I_m}{r}\cdot l \tag{5}$$

for the mutual force for a length l of two infinitely long parallel rectilinear conductors placed in empty space at a distance r apart, each carrying a current I_m, a *three dimensional electromagnetic system* of units is established. In this system, the charge unit $[Q_m]$ and the current unit $[I_m]$ are expressed in terms of the mechanical units and *dimensions must be attributed to the dielectric constant of empty space.* The electromagnetic units are summarized in Table II.

Each of the three-dimensional systems is logically sound and internally consistent, but in each case electrical and magnetic units are in general not expressible as products of integer powers of the fundamental units. Further, there is something rather "artificial" in taking the dielectric constant of empty space as dimensionless and then attributing dimensions to the permeability in the electrostatic system or in making the opposite choice in the electromagnetic system. It is also disturbing to some physicists to express electric charge in two different ways; for example, $[Q_s] = \mathrm{cm}^{3/2}\cdot \mathrm{g}^{1/2}\cdot \mathrm{sec}^{-1}$ and $[Q_m] = \mathrm{cm}^{1/2}\cdot \mathrm{g}^{1/2}\cdot \mathrm{sec}^{-2}$. One might ask facetiously whether an electron is one thing when measured in emu and quite a different thing when measured in esu!

As mentioned earlier, it has been recommended by various international commissions that electrical and magnetic quantities can be handled more conveniently by employing a "four-dimensional" system with one

TABLE I. Systems of Mechanical Units

Quantity	Defining Relation	MKS Units	CGS Units
Length	Fundamental	1 m	1 cm
Mass	Fundamental	1 kg	1 g
Time	Fundamental	1 sec	1 sec
Force	$F=Ma$	1 newton = 1 m · kg · sec^{-2}	1 dyne = 1 cm · g · sec^{-2}
Work	$W=FS$	1 joule = 1 nt · m = 1 m^2 · kg · sec^{-2}	1 erg = 1 dyne · cm = 1 cm^2 · g · sec^{-2}
Power	$P=W/t$	1 watt = 1 joule · sec^{-1} = 1 m^2 · kg · sec^{-3}	1 erg · sec^{-1} = 1 cm^2 · g · sec^{-3}

TABLE II. Three-Dimensional CGS Systems of Electrical and Magnetic Units*

	Quantity	Definition	Unit
Mechanical Units	Length	Fundamental	1 cm
	Mass	Fundamental	1 g
	Time	Fundamental	1 sec
Electrostatic Units	Dielectric Constant ϵ_0'	Dimensionless	1
	Charge Q_s	$F=\frac{Q_2Q_1}{\epsilon_0' r^2}$	1 statcoul = 1 $\text{dyne}^{\frac{1}{2}}\cdot\text{cm}$ = 1 $\text{cm}^{\frac{3}{2}}\cdot\text{g}^{\frac{1}{2}}\cdot\text{sec}^{-1}$
	Current I_s	$I=Q/t$	1 statamp = 1 $\text{statcoul}\cdot\text{sec}^{-1}$ = 1 $\text{cm}^{\frac{3}{2}}\cdot\text{g}^{\frac{1}{2}}\cdot\text{sec}^{-2}$
	Permeability μ_0'	$F=\mu_0'\cdot\frac{2I_1I_2}{r}\cdot l$	$\frac{1}{(2.9979)^2}\times 10^{-2}\,\text{cm}^{-2}\cdot\text{sec}^2$
	Magnetic Flux Density B_s	$F=BQV=BIl$	1 $\text{dyne}\cdot\text{statamp}^{-1}\cdot\text{cm}^{-1}$ = 1 $\text{cm}^{-\frac{1}{2}}\cdot\text{g}^{\frac{1}{2}}\cdot\text{sec}^{-1}$
	Electric Intensity E_s	$E=F/Q$	1 $\text{dyne}\cdot\text{statcoul}^{-1}$ = 1 $\text{cm}^{-\frac{1}{2}}\cdot\text{g}^{\frac{1}{2}}\cdot\text{sec}^{-1}$
Electromagnetic Units	Permeability μ_0'	Dimensionless	1
	Current I_m	$F=\mu_0'\cdot\frac{2I_1I_2}{r}\cdot l$	1 abamp = 1 $\text{dyne}^{\frac{1}{2}}$ = 1 $\text{cm}^{\frac{1}{2}}\cdot\text{g}^{\frac{1}{2}}\cdot\text{sec}^{-1}$
	Charge Q_m	$Q=I\cdot t$	1 abcoul = 1 $\text{abamp}\cdot\text{sec}$ = 1 $\text{cm}^{\frac{1}{2}}\cdot\text{g}^{\frac{1}{2}}$
	Dielectric Constant ϵ_0'	$F=\frac{Q_1Q_2}{\epsilon_0' r^2}$	1 $\text{abcoul}^2\cdot\text{dyne}^{-1}\cdot\text{cm}^{-2}=\frac{1}{(2.9979)^2}\times 10^{-20}\,\text{cm}^{-2}\cdot\text{sec}^2$
	Magnetic Flux Density B_m	$F=BIl$	1 gauss = 1 $\text{dyne}\cdot\text{abamp}^{-1}\cdot\text{cm}^{-1}$ = 1 $\text{cm}^{-\frac{1}{2}}\cdot\text{g}^{\frac{1}{2}}\cdot\text{sec}^{-1}$
	Electric Intensity E_m	$E=F/Q$	1 $\text{dyne}\cdot\text{abcoul}^{-1}$ = 1 $\text{cm}^{\frac{1}{2}}\cdot\text{g}^{\frac{1}{2}}\cdot\text{sec}^{-2}$

* The units shown here are *nonrational* units; the corresponding rational units can be obtained by the method described in the text. It should also be noted that corresponding three-dimensional MKS systems could be set up.

The Gaussian system of units, which has been widely used by physicists, is a "mixed system". In the Gaussian system the electrostatic electrical units and the electromagnetic magnetic units are employed, and the speed of light appears in relationships between the two. Thus, for the nonrational Gaussian system, Maxwell's equations assume the form

$$c\cdot\text{Curl}\ \mathbf{H}'_m = 4\pi\mathbf{J}_s + \frac{\partial \mathbf{D}'_s}{\partial t} \qquad \text{Div}\ \mathbf{D}'_s = 4\pi\rho_s$$

$$c\cdot\text{Curl}\ \mathbf{E}_s = -\frac{\partial \mathbf{B}_m}{\partial t} \qquad \text{Div}\ \mathbf{B}_m = 0$$

where for empty space $\mathbf{D}'_s = \mathbf{E}_s$ and $\mathbf{B}_m = \mathbf{H}'_m$.

fundamental electrical quantity in addition to the three fundamental mechanical quantities; current has been chosen as the fourth fundamental quantity. The ampere is chosen as the fourth fundamental unit in the MKSA system and is defined as follows:

> The ampere is the current which, flowing in each of two parallel infinitely long rectilinear conductors of negligible cross section, placed in empty space one meter apart, produces between them a force of 2×10^{-7} newton per meter length.

In the four-dimensional MKSA system, the units of all terms in equations corresponding to (4) and (5) have been selected and hence these equations become

$$F = 9\times 10^9 \frac{Q_1\ Q_2}{r^2} \qquad (6)^*$$

and

$$F = 2\times 10^7 \frac{I_1 I_2}{r}\cdot l, \qquad (7)$$

* For brevity in the text, $10^{-7}c^2$ is replaced by 9×10^9. The precise value is used in Table III.

where $[F] = 1$ nt, $[Q] = 1$ coul $= 1$ amp·sec, $[I] = 1$ amp, and $[L] = 1$ m. In these equations the constants, "9×10^9" and "10^7", are *physical quantities* or *dimensional constants* with dimensions chosen in such a way as to make the equations "balance" dimensionally. Thus, the units "$\text{nt}\cdot\text{m}^2/\text{coul}^2$" must be associated with "9×10^9" and "nt/amp^2" must be associated with "10^{-7}", where 1 nt $= 1$ $\text{kg}\cdot\text{m}/\text{sec}^2$.

In order to indicate that these coefficients are physical quantities, it is customary to express equation (6) in terms of a quantity called the dielectric constant of free space and (7) in terms of a constant called the permeability of free space. This can be done in the following way:

$$\left.\begin{aligned} F &= \frac{Q_1Q_2}{\epsilon_0' r^2} \\ F &= \mu_0'\cdot\frac{2I_1I_2}{r}\cdot l. \end{aligned}\right\}\text{Nonrational Form} \qquad (8)$$

These equations serve to define ϵ_0', the "nonrational" dielectric constant of empty space, and μ_0', the "nonrational" permeability of empty space, the numerical or "definitional factor" being unity in each case. When ϵ_0' and μ_0' as defined by (8) are employed, the "nonrational" or "unrationalized" form of the MKSA system results. When this form is used, Maxwell's equations have the form:

$$\text{Curl } \mathbf{H}' = 4\pi\mathbf{J} + \frac{\partial \mathbf{D}'}{\partial t} \qquad \text{Div } \mathbf{D}' = 4\pi\rho$$

$$\text{Curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Div } \mathbf{B} = 0$$

where $\mathbf{D}' = \epsilon_0'\mathbf{E}$ and $\mathbf{B} = \mu_0'\mathbf{H}'$ for empty space.

By writing equations (6) and (7) in a slightly different form by introducing a "definitional factor" of $1/4\pi$, one obtains

$$\left.\begin{aligned} F &= \frac{Q_1Q_2}{4\pi\epsilon_0 r^2} \\ F &= \frac{\mu_0}{4\pi}\,\frac{2I_1I_2}{r}\cdot l = \frac{\mu_0 I_1 I_2}{2\pi r}\cdot l \end{aligned}\right\}\text{Rational Form} \qquad (9)$$

where ϵ_0 is called the "rational" dielectric constant of empty space and μ_0 is called the "rational" permeability of empty space. When ϵ_0 and μ_0 as defined by (9) are employed, the "rational" or "rationalized" form of the MKSA system is obtained. When rational units are employed, Maxwell's equations assume the simpler form:

$$\text{Curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{Div } \mathbf{D} = \rho$$

$$\text{Curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Div } \mathbf{B} = 0$$

where $\mathbf{D} = \epsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$ for empty space. It is the rational form of the MKSA system of units that will probably be considered for adoption in 1954.

In addition to length, mass, time, and electric current, two other fundamental quantities are needed. In thermodynamics, *temperature* is considered as fundamental. The adopted temperature unit is the Celsius degree. The Celsius scale is for practical purposes the familiar centigrade scale. The zero on the Celsius scale is 0.01 Celsius degree below the triple-point of water; the degrees are equal in size to Kelvin degrees. The triple-point of water is 273.16°K or 273.17°K; the final value has still to be adopted by a future Conference of Weights and Measures. In optics, it is convenient to introduce *luminous intensity* as a fundamental quantity. The fundamental unit is the "candela" (cd), which has such a magnitude that the luminance of a black-body radiator at the freezing point of platinum is 60 cd per cm^2.

Thus, under present recommendations six physical quantities are regarded as fundamental: *length, mass, time, electric current, temperature,* and *luminous intensity.* The fundamental units are: the *Meter,* the *Kilogram,* the *Second,* the *Ampere,* the *Celsius Degree,* and the *Candela.* One of the remaining questions concerning units to be discussed at the 1954 meeting is the adoption of the rationalized form of the MKSA system for general use.

Comments and suggestions concerning the position to be taken by the American delegates to the general assembly of the International Union of Pure and Applied Physics and the American representative to the Commission of Symbols, Units and Nomenclature will be welcomed by the U.S.A. National Committee. Such communications should be transmitted to Professor J. C. Slater, Chairman USANC, Department of Physics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts.

In closing, the writer should like to express the hope that no change will be made in the time-honored and long established units of liquid volume. A *metric jigger* and a *rationalized fifth* might bring added confusion to a troubled world and would at best only form parts of a new incoherent system.

TABLE III. MKSA System of Electrical and Magnetic Units*

Quantity	Defining Relation	Units
Length	Fundamental	1 m
Mass	Fundamental	1 kg
Time	Fundamental	1 sec
Current I	Fundamental	1 amp
Charge Q	$Q = It$	1 coulomb = 1 sec · amp
Permeability μ_0	$F = \frac{\mu_0}{4\pi}\cdot\frac{2I_1I_2}{r}\cdot l$	$4\pi\times10^{-7}$ nt·amp^{-2} = 1.25664×10^{-6} m · kg · sec^{-2} · amp^{-2}
Dielectric Constant ϵ_0	$F = \frac{Q_1Q_2}{4\pi\epsilon_0 r^2}$	$\frac{1}{4\pi\cdot10^{-7}\cdot c^2}$ coul^2·nt^{-1}·m^{-2} = 8.85434×10^{-12} m^{-3} · kg^{-1} · sec^4 · amp^2
Magnetic Flux Density B	$F = BIl = BQV$	1 weber·m^{-2} = 1 kg · sec^{-2} · amp^{-1}
Electric Intensity E	$E = \frac{F}{Q}$	1 nt·coul^{-1} = 1 m · kg · sec^{-3} · amp^{-1}

* A similar four-dimensional system could be set up with CGS mechanical units and a current unit.