

Stiff-string theory: Richard Feynman on piano tuning

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In a letter to his piano tuner, the great theoretical physicist talks about how the nonzero stiffness of piano strings affects tuning, and he conjectures that piano tuners may need to pay more attention to ear-created harmonics.

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Theoretical physicist Richard Feynman (1918–88) was probably the post–World War II era’s most brilliant, influential, and iconoclastic figure in physics. He helped remake the field of quantum electrodynamics and was rewarded for that work with a share of the 1965 Nobel Prize in Physics. The problem-solving techniques that he invented permeated many areas of theoretical physics in the second half of the 20th century.

During World War II, Feynman was recruited to work on the Manhattan Project, first at Princeton University and later at Los Alamos, where he became the youngest group leader in the theoretical division. With the head of that division, Hans Bethe, he devised a formula for predicting the energy yield of a nuclear device. Feynman also took charge of the project’s primitive computing effort, using a mix of new calculating machines and human workers to process vast amounts of numerical data and obtain the information needed at Los Alamos.

He saw the first detonation of an atomic bomb, on 16 July 1945, at Alamogordo, New Mexico. All observers at the test site were issued dark glasses to protect their eyes in case the nuclear reaction was brighter than expected. Feynman didn’t wear the glasses. He reasoned that only UV radiation could harm his eyes, but UV would be stopped by ordinary glass. So he observed the detonation through the windshield of a truck. Unfortunately, the visible light, which did pass through the windshield, was very bright, and Feynman saw yellow blobs in his vision for days afterward. That episode proved to be a case of Feynman’s knowing too much for his own good, but fortunately, his eyes suffered no permanent damage. Though his initial reaction to the test was euphoric, he later felt anxiety about the force he had helped unleash.

While working on the Manhattan Project, Feynman became interested in the combination locks used to secure the file cabinets containing classified information. He discovered a way to determine the last two numbers of a three-number combination, which he described to me at a social gathering in 1985. Feynman had

noted that when the locks were open and lying on top of the filing cabinets, all of the tumblers would be properly aligned. He found that he could determine the last number of a combination by rotating the knob in one direction, one number at a time, until one of the tumblers was not aligned. By likewise rotating the knob in the other direction until a second tumbler was not aligned, he could determine the combination’s second number. Thus by nonchalantly fiddling with a lock as he visited with friends, he could determine the last two numbers of its combination. Now all he had to do was to determine the first number of the combination, which he did by trying all of the possible numbers. There were only about 30, so it didn’t take long. Scientists would open their file cabinets in the morning and find inside a note saying “Guess Who?” Everyone knew it was Feynman, but he never got into trouble for his playful antics. In fact, his skills were often utilized to open a file cabinet when data were needed and the combination was unavailable. On those occasions, Feynman said, he would request that everyone leave the room so they would not learn his secret. He claimed that he could open a lock in a couple of minutes, but he would read a comic book or do something else for another 15 minutes so



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that his friends would think safecracking was difficult.

Feynman's stature among physicists transcended even his sizable contributions to the field. His colleagues universally admired his bold and colorful personality unencumbered by false dignity or notions of excessive self-importance. He was a master calculator who could create a dramatic impression in a group of scientists by slashing through a difficult numerical problem. His intellectual achievements became a part of the scenery of modern science. Feynman diagrams, Feynman path integrals, and Feynman rules joined Feynman stories in the everyday conversation of physicists. His colleagues may have envied his flashes of inspiration, but they surely admired his faith in nature's simple truths, skepticism about official wisdom, and impatience with mediocrity.

The following is a transcript of a handwritten letter from

July 3, 1961

Dear Mr. McQuigg

I figured out the effect of wire stiffness on the vibration frequency of strings. The mathematical formula is [note 1]

$$\text{true frequency} = f \left(1 + \frac{\pi}{2} \frac{EA^2\mu}{T^2} f^2 \right),$$

where f is the frequency you would get forgetting about stiffness (for fundamental

$$f = \frac{2}{l} \sqrt{\frac{T}{\mu}}$$

[note 2] for string of length l , T is the tension in the string, A is the area of the string cross-section, E is Young's modulus of steel (measures the stiffness of the wire), μ is the weight of wire per unit length = 7.80 grams $\times A$ in squ. centimeters for steel. I have worked this out roughly for steel wires—not for the weighted bass strings. It says that the frequency is shifted by

$$\frac{1}{20} \text{ cent} \cdot \frac{(\text{diameter of string in mm})^6}{(\text{tension}/150 \text{ lbs})^2} \cdot \left(\frac{\text{frequency}}{100 \text{ Hz}} \right)^2$$

That is for middle C_4 , say, supposing $T = 150$ lbs and the wire diameter is 1 mm (.04 inches), the fundamental 261 is shifted by $1/20 (261/100)^2$ cents ≈ 0.35 cents, not much [note 3]. But the 3rd harmonic, used to match the fifth—ultimately with the string G_5 an octave above at 784 cycles (that is 3×261) the effect is $1/20 (784/100)^2$ cents ≈ 3 cents—still not much.

But suppose we use this G_5 to tune still higher up—to C_6 , C_7 and finally to G_7 at $3135 = 261 \times 2 \times 2 \times 3$. Then it is complicated to figure out because the diameter of the higher strings is less and the tension is smaller so it depends a bit on just how you do the tuning (I mean which string you beat with which). But if we forget about these variations in wire diameter and tension, our G_7 will be higher than it should be ideally by $1/20 (3135/100)^2$ [cents] ≈ 50 cents or half of a "semitone" (or whatever you call one note = 100 cents. 1200 cents = octave).

The effect may be somewhat less than this (1) because of the wire gauges decreasing near the top of the scale, (2) because you start at $A_4 = 440$ rather than C_4 , so the starting note is off by $1/20 (440/100)^2 \approx 1$ cent to begin with, but that we tune against a fork, so our entire shift is only $50 - 1 = 49$ cents. (Actually my arithmetic is not that accurate.) If you can send

Feynman to his piano tuner, Howard McQuigg (1916–2002) of Monrovia, California: McQuigg and I were neighbors in the 1970s, and over time we became close friends. During one of our many visits, he shared Feynman's letter with me; the online version of this article includes a copy of the original letter. As far as I know, Feynman never formally learned piano tuning, but he did own a piano, and he and McQuigg had apparently conversed previously on the subject of pianos and piano tuning. I think that had he wanted to, Feynman would have been a fine piano tuner.

In some places in the letter I have standardized notation or cleaned up equation presentation and punctuation. Occasional interjections direct the reader to explanatory notes that follow the letter. The box on page 48 explains the basics of piano tuning.

me an exact schedule of where you start going up and how to get, say, to G_7 (I mean by what beats) and what the wire gauges are (or better what their diameters are) I can get a better figure for how far it will be off.

I haven't tried to figure it for the bass strings.

Another effect that might alter things is the "give" of the sounding board. The end pins on the bridge are not absolutely rigid, of course, because they must move in order to move the sounding board—so the bridge is not precisely at the node of the string—but the node may be slightly behind the bridge (for high notes—for low notes the node could be even in front of the bridge if the sounding board is stiff enough). It is too hard for me to figure how big these effects would be.

Now comes the question: Suppose the effect is due to the string stiffness as I suggested—but suppose it was even stronger—say 200 cents for G_7 , and the same formula for all strings (tension and diameter constant). Question: Is it "better" to tune the piano by "ear" or by absolute frequencies? [note 4]

That depends on the theory of music. First of all, of course, [when?] playing with other instruments the character and pitches of the other instruments must be considered—but suppose the piano is alone.

Why does C_5 and G_5 , say, sound good together? The claim is made by some that the reason is that the harmonics are in unison (i.e., the 3rd harmonic of C_5 and the 2nd of G_5 are in unison, in the *not* equi-tempered scale). Ordinarily we take the 3rd harmonic of C_5 to be at frequency $3 \times f(C_5)$, where $f(C_5)$ is the fundamental of C_5 , say at 524. This should equal $2 \times f(G_5)$, so $f(G_5)$ should be $524 \times 3/2 = 786$. But suppose the 3rd harmonic, i.e., the vibration of the C_5 string with two extra nodes



is not at $3 \times f(C_5)$ but a little higher $(1/20 (3 \times 524/100)^2 - 1/20 (524/100)^2 = 10$ cents higher) [note 5]. In order that it sound well with the G_5 string we must arrange that $f(G_5)$ is so set that, not exactly $2 \times f(G_5)$, but rather, the true frequency of the G_5 string with one extra node



should be 524×3 raised by 10 cents. The net effect of all this is that $f(G_5)$ will have to be raised a few cents. (To be exact, $f(G_5)$ will be higher than $f(C_5)$ by $1/20 (786/100)^2 - 1/20 (524/100)^2 = 2$ cents roughly.) [note 6] Now will the notes

How to tune a piano

Musicians have developed their own jargon for naming musical notes and relations between them. Before reviewing the basics of piano tuning, I'll define some of that jargon with the help of the treble portion of the piano keyboard illustrated below.

Any two neighboring notes on the piano are said to differ by a semitone interval. Twelve semitones span an octave, and you can see in the illustration that the pattern of piano keys repeats after each octave. Two notes that differ by a number of octaves are given the same letter name; the octave is distinguished by a subscript. So, the interval separating C_4 and C_5 is an octave. Richard Feynman's letter discusses at length the relation between C_5 and G_7 , which are identified on the keyboard. The illustration also specifies F_4 (discussed later) and A_4 , the note sounded by the oboe to tune up an orchestra.

In all systems of tuning, every pitch may be derived from its relationship to a standard. In the case of piano tuning, the usual choice is to assign the frequency 440 Hz to the note A_4 . The frequencies of all the other notes are set by counting the beat rates that originate in upper, nearly coincident overtones when two notes are struck simultaneously. One might have thought that a piano could be tuned with the frequencies of any two notes related by simple whole-number ratios. Then all pairs of notes would be separated by "pure intervals," and one wouldn't need to worry about beat counting. It is mathematically impossible, however, to have only pure intervals in a standard 13-note-per-octave keyboard. Some of the intervals must be altered, which results in beating. Those altered tunings are referred to as temperaments.

Equal temperament is a system of tuning keyboard instruments in which the frequency ratio for any two notes separated by a semitone is $2^{1/12}$. Rather different, unequal temperaments are sometimes used for historical reasons. All temperaments are modified in piano tuning because the steel strings have nonzero stiffness, which causes the overtones to be higher in frequency than for simple harmonics; the effect is called inharmonicity. To quantify small frequency changes, piano tuners divide the semitone interval into 100 cents. The frequency ratio of two notes that differ by c cents is thus $2^{c/1200}$. As a result of inharmonicity, all keyboard intervals are stretched in frequency. Variations from equal temperament are almost negligible in the middle of the keyboard, but for a small, aurally tuned piano they rise to about 30 cents sharp at the treble end and about 30 cents flat at the bass. Larger pianos generally have less stretch, but it is always present.

Most piano tuners nowadays, including me, regularly use an electronic tuning device. The ETD makes the tuning process sim-

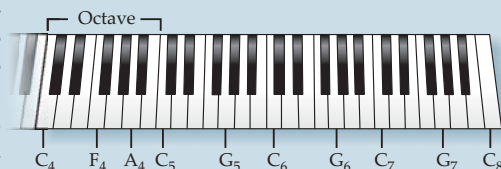
pler and less demanding on the ears. It usually produces good results, but we sometimes need to make corrections to render the tuning aurally acceptable. On the other hand, an ETD can detect minor flaws in an aural tuning. There are piano technicians with strong preferences on each side of the aural versus ETD divide. Aural tuning uses the ear—the ultimate judge of what sounds good. But human fatigue can make it difficult to tune in a noisy environment or to duplicate results. The ETD works well in noisy environments, never gets tired, and makes duplication easy. Generally, one method is strong where the other is weak, and many tuners prefer to use the best of both.

Aurally tuning a piano consists of three steps. The first is to establish the proper pitch for one note, usually A_4 . Next is to tune the temperament octave, usually the octave between F_3 and F_4 (F_3 , an octave below F_4 , is not in the portion of the piano keyboard illustrated here). At last, using the temperament octave as a standard, one can tune the rest of the piano. Note that when a piano key is pressed, two or three strings inside the piano are struck. A piano technician needs to tune each string individually.

A simplified version of a tuning procedure goes something like this: Two of the three strings of each note in the temperament octave are muted so that only one will vibrate when the corresponding piano key is played. Then a 440-Hz tuning fork is sounded and the tension in the A_4 string is adjusted so that no beats are heard. Next, one tunes A_3 by playing A_3 and A_4 together and adjusting the tension in the A_3 string until no beats are heard. Because of inharmonicity, the octave interval A_3 – A_4 will be slightly wider than 1200 cents.

Next, the "major third" interval F_3 – A_3 is tuned. In equal temperament, the frequency ratio of notes separated by a major third is $2^{4/12}$ (1.2599), about 14 cents wider than the pure major-third interval of $5/4$. The result is that a piano tuner who plays F_3 and A_3 together will hear about 7 beats per second. The beat rate for major thirds increases as one goes up the keyboard; it doubles to 14 beats per second for the next octave F_4 – A_4 , doubles again for F_5 – A_5 , and so forth. The goal in setting the temperament octave is not to count theoretical beat rates exactly but to make them progress evenly through the octave. Tuners use various means to accomplish that goal.

After the temperament octave has been set and tested, it serves as the standard for tuning the remaining notes of the piano. One simply plays octaves and adjusts the tension in the untuned note until there are no beats. If the process is carried out carefully, the piano will sound good. In the end, tuning a piano is an art as well as a science.



sound well? They ought to—not only will the 3rd harmonic of C_5 and the 2nd harmonic of G_5 be in unison—but the 6th harmonic of C_5 and the 4th harmonic of G_5 will be too—etc., and this latter will be in unison with G_7 fundamental [note 7], as it ought if and only if G_7 is tuned high (by 48 cents relative to $f(C_5)$) as it will be by ear automatically.

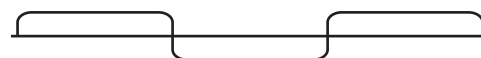
But another claim is that distortions of the ear create harmonics automatically—so even a pure tone of 524 with no natural harmonics (I mean those created by the instrument) and a pure tone of 786 sound well together because the 3rd harmonic created in the ear, or ear-brain system, of C_5 fits with the 2nd of 786. These harmonics created by the ear (if they exist at all) must be *exactly* three times frequency, etc. with no error—so the piano

should be tuned to absolute frequency.

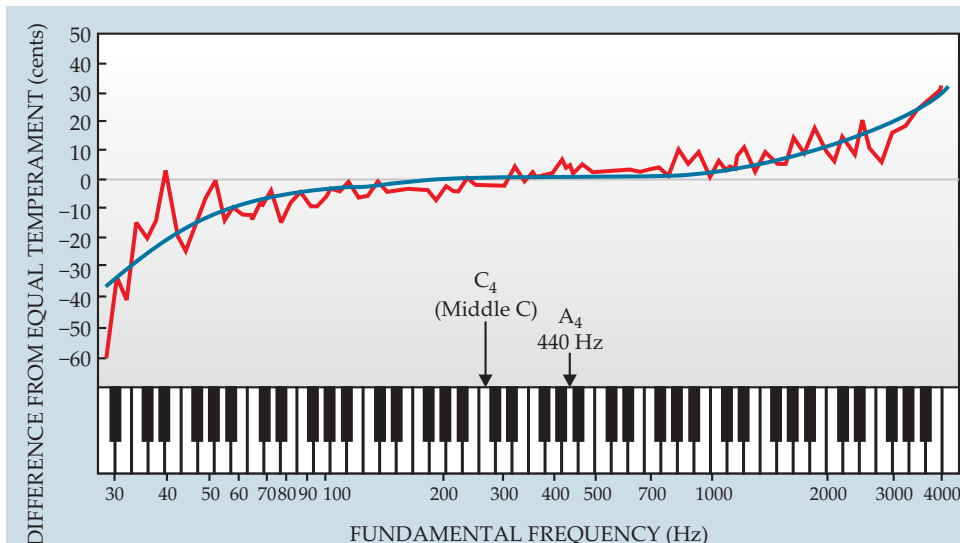
Why are the ear-created (or bad-amplifiers-etc-in-radio-and-phonograph-created) harmonics exact multiples? Suppose the pure wave in is



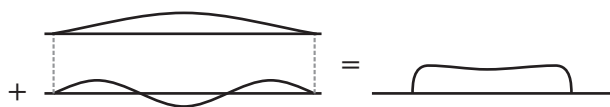
—it gets amplified and distorted, say, to



which is flatter tops, for example. But this flat top wave is equivalent to adding pure waves



Pianos sound better when they are tuned by ear slightly away from equal temperament, as shown in a 1961 study by Daniel Martin and W. Dixon Ward. The red curve shows the aural tuning of the technician used in the Martin and Ward work. The blue curve is a comparative standard that averages over many aurally tuned pianos. In equal temperament, the frequencies of two neighboring notes on the piano differ by 100 cents. (Adapted from D. W. Martin, W. D. Ward, *J. Acoust. Soc. Am.* **33**, 582, 1961.)

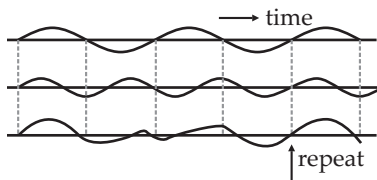


To work, these must be exactly 1:3. To put it differently, maybe the ear “likes” a perfectly repeating vibration—

so if C_5 is:

and if G_5 is:

together the ear gets:



which repeats exactly after 2 vibrations of C_5 and 3 vibrations of G_5 .

If this is the way things work—I mean [consonances?] sound good—then we ought to correct our “aural” piano tuning by appropriately lowering the high notes so they are closer to the mathematical frequencies.

Someday maybe I’ll do some experiments to find out.

Make an artificial “electric organ” for which each note played produces a mixture of pure tones, not at relative frequencies 1; 2.00; 3.00; 4.00; etc. as an ideal string, or a wind instrument like the voice does; but is badly offset—say like 1; 2.10; 3.18; 4.40; etc. by some formula and then set the 1st “octave” of this organ so that its fundamental is 2.10 (1st harmonic 4.40, etc.) [times] the note below, etc. Just as a “piano with an extreme error for string stiffness” and see what it sounds like. For consonant intervals like “octave,” “fifth,” etc. harmonics will be always in unison—yet the frequencies are not 1; 2; 3; 4; etc.

Nobody could sing with it, of course (voice harmonics are at 1; 2; 3; 4), and it might sound very strange—yet there is the bare possibility that regular chords on it like major triad would sound well enough. I’ll try to talk some student into trying it.

Anyway we would finally know whether it’s the jarring harmonics or the distortions from perfect repetitions of [waves?] in the ear which is the basis of our feeling of dissonance. I suspect it is the latter and you piano tuners have been mistuning pianos since they were invented. But who knows, I must try the experiment.

Yours,

Dick Feynman ■

Explanatory notes

1. Feynman’s formula for the “true” frequency of a string with finite stiffness may be written

$$\text{true frequency} = f \left(1 + \frac{B}{2} \right),$$

where $B = \pi E A^2 \mu f^2 / T^2$ is the inharmonicity coefficient; the other terms are defined in the letter.

To evaluate B , I use the following data, obtained from the Mapes Piano String Co. My values differ slightly from those cited by Feynman. With diameter of string, 1 mm; string tension, 150 lb (667 N); density of steel, 7840 kg/m³; frequency, 261.6 Hz, corresponding to the middle C of a piano; and Young’s modulus for steel, 2.07×10^{11} N/m²; I find $B = 0.00038$.

As usual in piano work, especially for notes in the middle of the keyboard, $B \ll 1$. It gets larger at both ends of the keyboard, which results in a greater frequency stretch.

2. The equation should read

$$f = \frac{1}{2l} \sqrt{\frac{T}{\mu}}.$$

That is, the “2” should be in the denominator, not the numerator as in the letter.

3. By definition, the measure of a frequency ratio in cents c for two frequencies f_1 and f_2 is given by $f_2/f_1 = 2^{c/1200}$. For additional details, see the box on page 48.

4. The answer is that a piano tuned by ear sounds better. In 1961 Daniel Martin and W. Dixon Ward reported that listeners unequivocally rejected strict equal temperament tuning in favor of tuning by ear. As the figure on this page shows, tuning a piano by ear results in a “stretched” scale in which the upper tones are higher and the lower tones lower than in equal temperament. For a typical small piano, the highest note is about 30 cents sharp, and the lowest note is about 30 cents flat.

5. The first term in Feynman’s difference expression correctly describes the ratio of the third harmonic to treble the fundamental. The subtracted term seems to have been added in the letter as an afterthought.

6. This is clearly a slip of the pen: The fundamental frequency for G_5 is about 3/2 that of C_5 .

7. Because of inharmonicity, Feynman’s statement is not correct for a real piano. The higher partial frequencies of a real piano string are stretched more than the lower ones. Therefore, even if the 3rd partial of C_5 is in unison with the 2nd partial of G_5 , the 6th partial of C_5 will still beat against the 4th partial of G_5 .