

Medieval dynamics

Edith Dudley Sylla

Thomas Bradwardine's 14th-century dynamical law may not be well known today, but it greatly influenced European scholars through the late Middle Ages and into the Renaissance.

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In his *Specimen dynamicum* of 1695, Gottfried Wilhelm Leibniz, co-inventor of the calculus, proposed and named a “new science of dynamics” that would include forces or the causes of motion as well as their effects.¹ But even if not given that name, a science of dynamics had been in existence since the 14th century.² Its foundations were laid in 1328, when Thomas Bradwardine's *De proportionibus velocitatum in motibus* (*On the Proportions of Velocities in Motions*) presented a mathematical law linking any velocity to the proportion of motive to resistive forces causing it.

Aristotle, in Book VII of his *Physics*, had discussed a few cases relating forces, the bodies moved, distances, and times. He said, for instance, that if a given force moves a resisting body over a certain distance in a given time, then in the same time the same force will move half the resistance over twice the distance, “for thus the rules of proportion will be observed.” But Aristotle did not express the relationship in general terms. More importantly, he restricted the inferences that could be made, and said, for instance, “It does not follow that, if a given motive power causes a certain amount of motion, half that power will cause motion either of any particular amount or in any length of time: otherwise one man might move a ship.”

It is frequently said that Aristotle took a qualitative and common-sense approach to natural philosophy, not a mathematical one. Bradwardine, in contrast, emphasized that he was taking a mathematical approach (see the box on page 52). By expressing his law not in terms of forces or resistances separately but in terms of “proportions of greater inequality” (that is, proportions of a larger quantity to a smaller one), he

could rightfully assert that his law could deal with all variations in such proportions and with all velocities.

From the moment Bradwardine's work became known, the “proportions of velocities in motions”—the mathematical science of motion—found a place in university curricula. And it kept that place for the next two centuries.³ At Oxford University, the line of inquiry into dynamics that Bradwardine had opened was pursued through the mid-14th century by the likes of William Heytesbury, John Dumbleton, and Richard Swineshead, who are together referred to as the Merton school or the Oxford Calculators.⁴

At the University of Paris, works *de proportionibus* were written by Albert of Saxony and Nicole Oresme (depicted in figure 1) in the 14th century. In 1509 in Paris, Álvaro Tomás of Lisbon published his large and impressive *Liber de triplici motu proportionibus annexis magistri Alvari Thome Ulixbonensis philosophicas Suiseth calculationes ex parte declarans* (*Book on the*



Figure 1. Nicole Oresme (1323–82) received his master of theology degree from the University of Paris in 1355. He is perhaps the most brilliant of the many European scholars who were influenced by Thomas Bradwardine's *De proportionibus velocitatum in motibus*. The original portrait is in the Bibliothèque Nationale, Paris.

The essential role of mathematics

Thomas Bradwardine's *De proportionibus velocitatum in motibus* (*On the Proportions of Velocities in Motions*), published in 1328, begins with a self-contained primer on the mathematics of proportions. As Bradwardine explains,⁶

Since each successive motion is proportionable to another with respect to speed, natural philosophy, which studies motion, ought not to ignore the proportion of motions and their speeds, and, because an understanding of this is both necessary and extremely difficult, nor has as yet been treated fully in any branch of philosophy, we have accordingly composed the following work on the subject. Since, moreover (as Boethius points out in Book I of his *Arithmetic*), it is agreed that whoever omits mathematical studies has destroyed the whole of philosophic knowledge, we have commenced by setting forth the mathematics needed for the task at hand, in order to make the subject easier and more accessible to the student.

Triple Motion, Together with Proportions, by Álvaro Tomás of Lisbon, *Explaining in Part the Philosophical Calculations of Swineshead*). Fifteen years later, an abbreviated version of Bradwardine's *De proportionibus* was printed, along with short texts on logic, as part of handbooks for undergraduates (*Libelli sophistarum*) at Oxford and Cambridge universities. In the late 17th century, Leibniz praised Swineshead, called "the Calculator," for beginning to introduce mathematics into scholastic philosophy.⁵

Bradwardine's doctrine on the proportions of velocities in motions was a significant achievement. According to Bradwardine, a simple and universal mathematical law describes dynamics: Velocities vary as the proportions of motive to resistive forces causing them. In stating that law, Bradwardine overcame the limitations of what had traditionally been understood as the Aristotelian position on dynamics. At the same time, Bradwardine could plausibly claim that his function represented what Aristotle really intended. In fact, his view agreed with Aristotle's in cases in which one starts with a force double the resistance and when the force is then doubled; the proportion is doubled, and so the velocity will be doubled in both Aristotle's view and Bradwardine's. Moreover, the compendium on the mathematics of proportions that Bradwardine placed at the start of *De proportionibus* proved its value as a core text of the university curriculum. Given those factors, it is no wonder that Bradwardine's *De proportionibus* had such a Europe-wide and long-lasting influence.

Compounding proportions

Mathematical physics expresses physical relationships in mathematical terms. To contribute to mathematical physics, one may gather new empirical information about the physical world; develop or improve the conceptual apparatus through which mathematics is to be applied to the world, by defining, for instance, what is meant by "force" or "velocity"; invent new mathematical tools such as the calculus; or show how to apply previously existing mathematics in a new way. Bradwardine's originality in founding medieval dynamics fell into the last of those categories: He showed how the mathematics of proportions, which already existed in the discipline of music—the mathematical science that along with

arithmetic, geometry, and astronomy formed the *quadrivium* of mathematical sciences studied in medieval schools and universities—could be applied to dynamics.

For Bradwardine, a proportion is a relation with respect to size between two quantities that have the same dimensions.⁶ One can have a proportion between two quantities a and b , or one can think of a series of terms a, b, c , and so forth, with proportions between successive terms. On that understanding, a proportion is not an indicated division or fraction but rather a relation in which the two quantities remain two. For example, when plucked together, two similar strings under similar tension with lengths in a proportion of 2 to 1 produce sounds in the consonance of an octave. In their proportion to each other, the two strings remain two.

Now consider three quantities of the same kind arranged to decrease monotonically—for example, the lengths of strings sounding in a musical chord. In Bradwardine's language, one would say that the proportion of the first to the third is "compounded" (*composita*) of the proportion of the first to the second and the proportion of the second to the third. Figure 2 shows how medieval music theorists illustrated the compounding of musical intervals. In one application, an octave corresponding to a ratio of 12 to 6 is shown to be equal to two musical fourths corresponding to proportions of 12 to 9 and 8 to 6, plus a musical tone corresponding to the proportion of 9 to 8.

The Latin translations of Euclid's *Elements* available in the Middle Ages, in particular the Campanus edition that Bradwardine used, contained no definition of compounding proportions. But compounding was used in the proof of Proposition 23 of Book VI of the *Elements*. There, in Thomas Heath's translation, Euclid says that "the proportion of K to M is compounded out of the proportion of K to L and of [the proportion of] L to M ."⁷ Lacking a definition of compounding, Bradwardine used Euclid's definitions of *duplicata* and *triplicata* from Book V of the *Elements*, in which Euclid wrote, "If there are three continually proportional quantities, the proportion of the first to the third is said to be the proportion of the first to the second duplicated (*duplicata*)."⁸

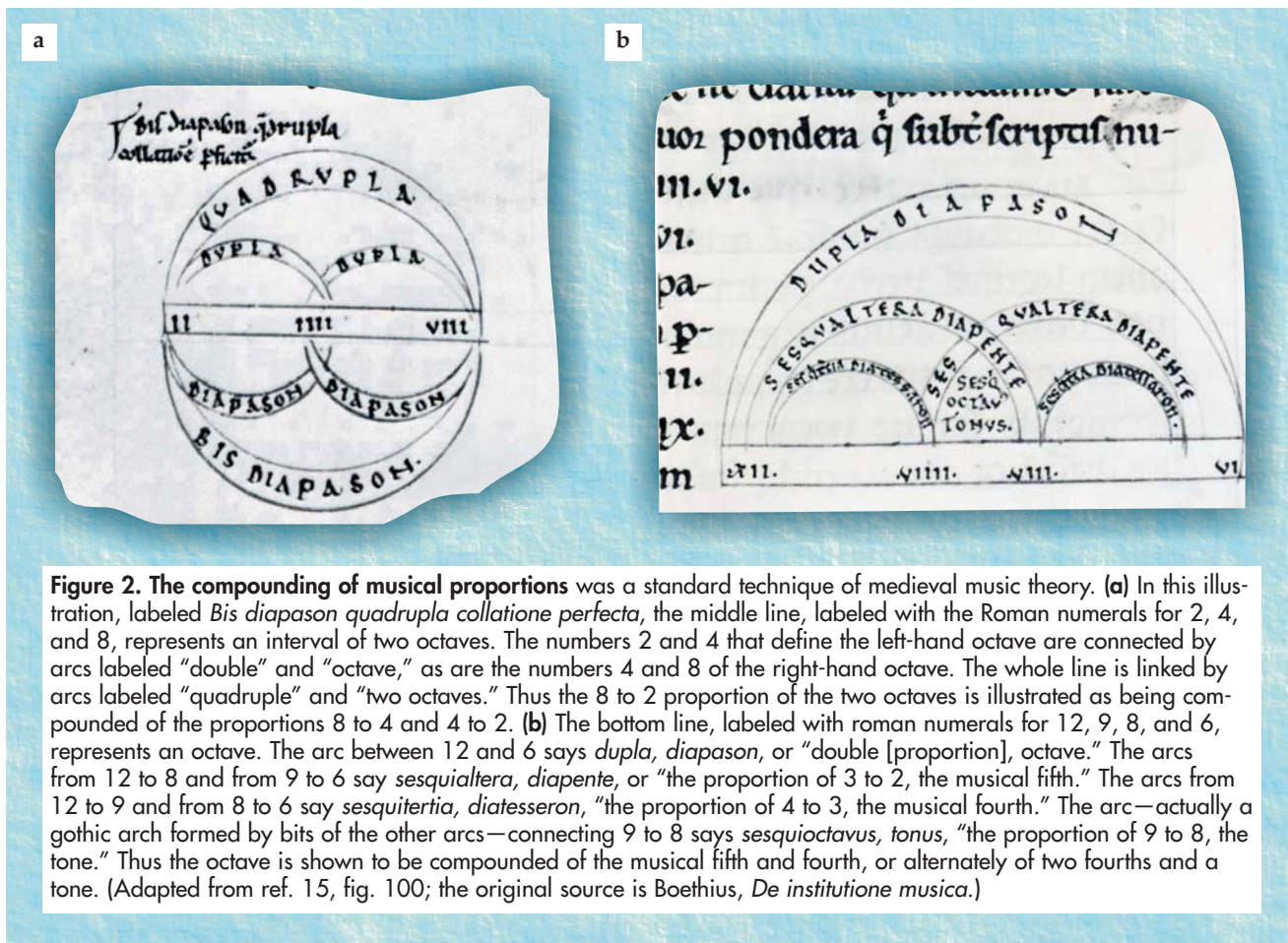
Bradwardine interpreted Euclid's "duplicate" (*duplicata*) as meaning "double," as is evident in the first theorem in *De proportionibus*:

If a proportion of greater inequality between a first and a second term is the same as that between the second and a third, the proportion of the first to the third will be precisely double (*dupla*) the proportion of the first to the second and of the second to the third.⁶

Similarly in his second theorem, which involves four, five, or more terms in continuous proportionality, Bradwardine says that the proportion of the first to the fourth term is triple (*trippla*) the proportion between any two successive terms, the proportion of the first to the fifth is quadruple (*quadrupla*), and so forth.

Bradwardine's way of handling proportions and their "addition" was more flexible and powerful than is sometimes recognized. It easily allowed for the compounding of unequal proportions, whether or not they could be expressed as proportions of integers. According to Bradwardine:

Given any two extreme terms with an intermediate term between them having a proportion to each extreme, the proportion of the first term to the third is compounded of (*composita ex*) the proportion of the first term to the second and of the proportion of the second term to the third.⁶



In applying this supposition to dynamics and using the mathematical foundation he had previously set out, Bradwardine could express his dynamic law in supremely simple form: The velocities in motions follow the proportions of forces to resistances. So, for instance, when the proportion of force to resistance is doubled or tripled, the velocity is doubled or tripled. Here it must be understood that, for example, double the proportion 3 to 1 is the proportion 9 to 1, and triple the proportion 3 to 1 is the proportion 27 to 1. On the other hand, doubling or tripling a velocity corresponds to multiplying by 2 or 3.

Beginning with Anneliese Maier,⁹ the first modern historian to understand what Bradwardine intended, scholars have represented Bradwardine’s law using logarithmic or exponential equations equivalent to $v = \log(F/R)$. That is, the velocity increases as the logarithm of the ratio of force to resistance. Figure 3 shows Bradwardine’s law and contrasts it with Aristotle’s view. Neither logarithms nor the mathematics of exponents had been developed in the 14th century, however, so medieval scholars did not think of Bradwardine’s law in such terms. Although we may think we understand Bradwardine’s function better by expressing it logarithmically or exponentially, there is an important reason not to explain it that way. Namely, the more modern formulation conceals the simplicity of Bradwardine’s law as expressed in his own terms. And that simplicity was one of the greatest reasons for its rapid adoption by Bradwardine’s contemporaries, who then proceeded to show in practice how, given one proportion of force to resistance and the velocity caused by it,

other velocities could be calculated for other proportions of force to resistance.

A problematic discontinuity

Bradwardine’s primary argument against the traditional Aristotelian position was fundamentally a mathematical argument: The Aristotelian view does not deal adequately with its boundary condition that the velocity vanishes when the force and resistance are equal. If taken to apply beyond the few examples he gives, Aristotle’s theory of the relations of movers, things moved, distances, and times predicts a non-vanishing velocity for all cases in which the force is greater than the resistance, but then a sudden halt when force equals resistance. As a result, motions would start and stop with a jerk and some low velocities would not correspond to any proportion of force to resistance. In figure 3, for example, the minimum nonvanishing Aristotelian velocity is 1.

But small velocities are known to occur. Moreover, Bradwardine argues, one can see that a small velocity increases in greater proportion than the force, an observation that also goes against the Aristotelian opinion. In making his argument, Bradwardine appeals to observations of men lifting weights and to what may happen in a weight-driven clock, but he does not check his view and alternatives to it against experiment.⁶

The sudden discontinuity when force equals resistance can be avoided if velocity depends on the excess of force over resistance. Bradwardine rejects that opinion on the grounds that it is inconsistent with experience. It implies, for instance,

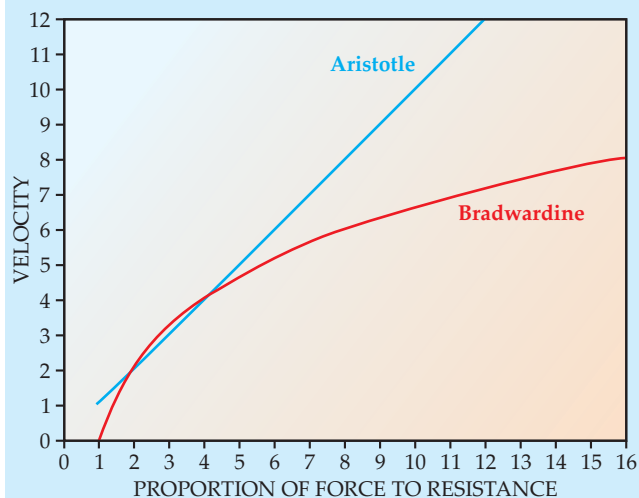


Figure 3. Bradwardine's law, that velocities in motions follow the proportions of forces to resistances, would be expressed as a logarithmic relation in modern language. The graph contrasts Bradwardine's dynamical law with Aristotle's linear relation when the force is greater than the resistance.

that a strong man will move a large object resisting him faster than a boy or a fly will move smaller objects, but boys and flies carrying small objects can move very fast.⁶ Having refuted the other possibilities, Bradwardine was left with his own theory.

In brief, Bradwardine's law appeared beautifully simple, given his approach to compounding proportions as addition, and it was mathematically superior to the Aristotelian theory because it avoided an implausible discontinuity in velocity as force comes to equal resistance. What remained was to show how it could be used in practice. In the years that followed Bradwardine's *De proportionibus*, many students and teachers studied the mathematical theory of proportions it expounded and drew implications from Bradwardine's law. John Dumbleton, in his mid-14th-century work *Summa logicae et philosophiae naturalis* (*Sum of Logic and of Natural Philosophy*), represented Bradwardine's function in terms of two parallel lines. As shown in figure 4, one line represented the "latitude of motion," a linear scale with zero at one end; the other line represented the "latitude of proportion," with the proportion of unity at one end and with equal intervals corresponding to equal proportions. By comparing the numbers on the line of motion to those opposite on the line of proportion, one can read off the velocity that results from a given proportion of force to resistance.

Swineshead, in his *Calculaciones* (*Calculations*, circa 1350), showed how to calculate other pairings of proportions of force to resistance with the corresponding velocities, given one such pairing.¹⁰ In Treatise XIV, Swineshead begins by indicating how the proportions of force to resistance increase. His first rule is

Whenever some power increases with respect to a constant resistance, as great a proportion is acquired with respect to that resistance as the power itself becomes greater, or in other words with the same meaning: it acquires as great a proportion with respect to itself as it acquires with respect to the resistance.

Swineshead's fourth rule is

Whenever some power increases or decreases with respect to two resistances, whether they are equal or unequal, but unchanging, the motion will increase or decrease with equal velocity with respect to either.

In other words, since in either case the added or subtracted proportion (as understood in Bradwardine's sense of compounding as addition) is the proportion of the new force to the old one, the same change in velocity will occur. In manuscripts of the *Calculaciones*, scribes or owners represented Swineshead's rules with numbers distributed along lines and connected by arcs just as musical intervals were represented in music manuscripts. Figure 5 gives examples.

On the basis of his rules, Swineshead builds up ever-more-complicated cases dealing not only with a single change but with continuous changes in the proportion of force to resistance over time and with the resulting continuous changes in velocity. He is especially interested in uniformly increasing or decreasing velocities. To analyze those cases, he first posits a homogeneous resisting medium whose resistance increases over time in such a way as to cause a uniformly decreasing velocity in a moving body. He then substitutes for that an unchanging but inhomogeneous medium that has, at every point, a resistance equal to the resistance that the first, changing medium had when the body reached the corresponding point. Swineshead could not represent analytically or mathematically how that resistance varies over distance; he could describe the variation only in the above, roundabout way.

Such cases led to paradoxes for Swineshead's and Bradwardine's views, paradoxes that also arise for the traditional Aristotelian position. Suppose a body is moving through a medium with varying resistance to the motion. The instantaneous velocity ascribed to the body using Bradwardine's function and the force and resistance being encountered at any instant (called the measure of velocity with respect to cause, or *tanquam penes causam*) may not be consistent with the velocity as measured by the distance traversed over time (called the measure of motion with respect to effect, or *tanquam penes effectum*). Tomás addresses the paradox explicitly in the *Liber de triplici motu* of 1509, along with problems

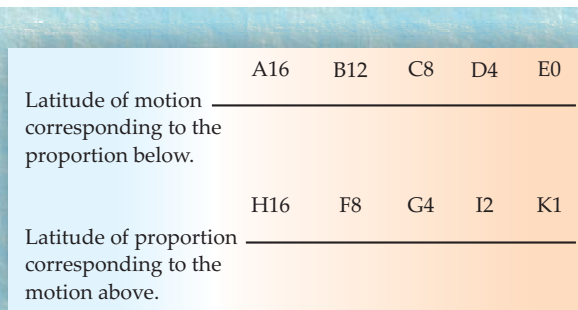


Figure 4. Latitudes of motion and proportion represent Bradwardine's law. The sketch here is based on an illustration from a mid-14th-century work by John Dumbleton. On this pair of latitude lines, the proportion of force to resistance H causes the velocity A, the proportion F causes the motion B, and so on. Evidently, each doubling in proportion contributes a fourth part of the velocity A. (Adapted from ref. 16.)



Figure 5. Swineshead's rules, as given in his *Calculaciones* (*Calculations*), could be expressed in terms of lines and arcs. In these drawings, Swineshead's rule 1 is represented at the left and rule 4 is represented fourth from left. The numbers for rule 4 should be 12, 6, 4, and 2. Note how similar these diagrams are to the music-theoretic diagrams in figure 2. (Adapted from ref. 15, fig. 256; the original manuscript is at the Biblioteca Universitaria in Pavia, Italy.)

that result when the resistance of a medium is supposed to increase abruptly and the interface between lesser and greater resistances is itself supposed to be moving, possibly trapping a moving body at the interface.¹¹

The scope of medieval dynamics

After establishing his dynamical law in *De proportionibus*, Bradwardine went on to discuss how to measure velocities of rotation of a spherical shell—obviously with the heavenly spheres in mind. Although he referred to the opinion that one ought to calculate something like a mean speed for rotating bodies, Bradwardine decided instead to measure motions of rotation by the distance traversed by the body's fastest moving point. As applied to the heavens, the decision between the alternative measures perhaps did not matter because it was often assumed that the usual relations of force, resistance, and velocity do not apply in the case of the heavenly spheres. The Oxford scholars who followed Bradwardine's program in dynamics largely also followed him in measuring the speed of irregularly moving bodies by the distance traversed by the fastest moving point.

Some scholars, however, argued that the speed of a rotating radius should be taken to equal the speed of its midpoint, or equivalently, half the speed of its outermost point. That result was a corollary of the so-called Merton mean speed theorem, more famous for its application to motion uniformly accelerated in time. In that case the theorem said that a uniformly accelerated motion will traverse a distance equal to the distance that would be traversed in the same time with a uniform velocity equal to that at the middle instant of the motion. At Paris, Oresme represented the theorem geometrically, and it was later used without attribution by Galileo Galilei.¹²

Oresme used Bradwardine's function to try to persuade Charles V and his courtiers to be skeptical of astrologers. Even admitting that the same sorts of earthly events may happen when the planets are in the same celestial configurations, Oresme reasoned on the basis of Bradwardine's function that planetary motions are most probably incommensurable, which means any given configuration of the planets in given positions in the heavens would never be repeated.¹³ To make

the case, he had to consider how many "proportions of proportions" could be represented by proportions of integers. The proportion of 8 to 1, for instance, is three halves to the proportion of 4 to 1 because the proportion of 4 to 1 is compounded of two proportions of 2 to 1 and the proportion of 8 to 1 is compounded of three proportions of 2 to 1. It is much more probable, Oresme said, that any two proportions of force to resistance will not be related in a proportion of integers than that they will be. To the argument that in heavenly motions the sorts of forces and resistances that apply to terrestrial motions are not applicable, Oresme simply indicated some other factors analogous to force and resistance would apply.¹³

In the two centuries during which Bradwardine's *De proportionibus* continued to be influential, scholars not only learned to work with the mathematics of proportions and not only discussed how to measure local motions, but they also studied limits, such as first and last instants of motion and maxima and minima of powers. Is there, for instance, a maximum weight that a man can carry or a minimum weight that he cannot carry?

More importantly, medieval dynamics was assumed to apply as much to motions of augmentation, such as growth of animals or the rarefaction of air, and to motions of alteration, such as heating and cooling, as to local motions. Scholars asked whether the rate of augmentation depends on the net quantity added or on a proportional increase. Would adding an inch to a seedling and to a tree be equal augmentations?

With regard to rates of heating, scholars asked whether the rate of heating with respect to effect depends only on the intensity of heat gained, and if not, whether it depends on changes in the intensity of the hottest part of the body only or whether it also depends on how much of the body is heated to varying degrees. Does the motion with respect to cause depend only on the degree of heat of the agent, or does it also depend on the extent of the agent or even on the quantity of heat within the agent? And is the resistance of the body being acted on measured by its degree or also by the size of the body acted on? Considering a hot body in contact with a cold one raised a theoretical problem because it was assumed

that when a hot body warms a cold one, the cold body simultaneously cools the hot body. How can each of the two bodies have a force greater than resistance, supposedly necessary to cause motion?

Compounding reinterpreted

It might be supposed that Bradwardine's dynamics was overthrown along with Aristotle's view upon adoption of the law of inertia. Like Aristotle, Bradwardine assumed that motion must have a cause or force producing it. But ironically, it was not the lack of empirical fit with new physics that led to the rejection of Bradwardinian dynamics; rather, it was the mathematics in which Bradwardine expressed his dynamics.

Bradwardine's law rested on understanding the compounding of proportions as addition. He had relied on Campanus of Novarra's translation of Euclid's *Elements* from the Arabic, which was the translation published by Erhard Ratdolt in Venice in 1482. Later editions based on Greek manuscripts and the new Latin translations published by Batolomeo Zamberti (1505) and Federico Commandino (1572) introduced into Euclid's Book VI a definition of compounding proportions that Renaissance mathematicians mistakenly believed to be authentically Euclidean: In Heath's translation, "a ratio is said to be compounded of ratios when the sizes of the ratios multiplied together make some [?] ratio, or size."⁷ As Heath remarks, "It is beyond doubt that this definition of ratio is interpolated."⁷ Nevertheless, the interpolated definition started a landslide in which the understanding of compounding as multiplication rapidly buried the view of compounding as addition. Once compounding came to be understood as multiplication, and proportions, now rebap-

tized as ratios, came to be understood as indicated divisions of numbers, the life of Bradwardine's dynamical law of motion was at an end—more than a century and a half before it might have been disproved on physical grounds with the acceptance of Newton's laws.

In the 16th and 17th centuries, mathematicians, who identified ratios with fractions or rational numbers, began to assert that it is an abuse in mathematics to say that when a proportion is compounded with itself, it is "double" its original.¹⁴ For this reason, Bradwardine's law was not even a contender by the time of Galileo, who argued against traditional Aristotelians rather than against the dynamics of the followers of Bradwardine, Swineshead, Oresme, and Tomás. That neglect, however, was more or less an accident of the history of mathematics. After all, the definition of compounding as multiplication was not authentically Euclidean, and the understanding of compounding as addition was a viable alternative approach, especially when proportions were applied to quantities of all kinds and not only to numbers.

The development of mathematics in the 16th century, however progressive from its own point of view, need not detract from our admiration for what Bradwardine and other late medieval university scholars attempted to do in dynamics two centuries earlier. They deserve to be recognized for creating and sustaining a program of study devoted to a mathematical science of the causes and effects of motions—that is, to founding a mathematical science of dynamics.

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