Chaotic Dynamics and the Origin of Statistical Laws

Chaotic dynamics in real systems does not provide finite relaxation time to equilibrium or fast decay of fluctuations, and chaotic systems are not completely random in the sense originally postulated for statistical systems. These properties may require rethinking some of the fundamental assumptions of thermodynamics.

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The problem of the foundation of statistical physics emerged with the derivation by Ludwig Boltzmann[1] of a kinetic equation for a gas of molecules that required monotonic growth of entropy. Boltzmann's theory leads to modern thermodynamics, and, for example, to the impossibility of gas spontaneously gathering in one part of a container in the absence of external forces. This result, known as the H-theorem, met with strong contemporary opposition, especially from mathematician Ernst Zermelo.

The foundation of statistical laws, as understood by the modern physics community, encompasses three principal aspects: the origin of statistical laws from deterministic dynamical equations, the conditions of applicability of different statistical approximations, and criteria for the transition from deterministic to statistical behavior. The discovery of chaos in dynamical systems makes it necessary to reconsider our views on each of these aspects, with potentially significant consequences.

Zermelo's objections[2] to Boltzmann's H-theorem were based on a nonrigorous application of a rigorous result of Henri Poincaré, the so-called Poincaré recurrence theorem. Another criticism came from Boltzmann's teacher, Josef Loschmidt, who pointed out that, due to the reversibility of dynamical equations, we can reverse all trajectories of a system and thereby return all system characteristics to their initial values. In particular, this reversal should occur with the entropy, which would then decrease on time reversal, in contradiction to Boltzmann's results.

The drastic difference between statistical behavior, based on probabilistic laws, and pure deterministic behavior, based on Hamiltonian dynamics, leads us to the following fundamental questions: Is the statistical description purely a technical (or instrumental) way to describe the dynamics, or a reflection of naturally existing random irregularities? Can we, starting from Hamiltonian dynamics (as Boltzmann actually did), explain the appearance of thermodynamics (statistical physics, H-theorem, entropy growth, and so on)? Going further, we may ask: Is the fact that it is impossible for molecules of gas uniformly distributed in a chamber to gather in full, or almost in full, in a part of the chamber without the help of external forces a real and absolute property of nature, or is it a result of the convenient use of some theoretical approximation methods?

After Boltzmann's death in 1906, Paul and Tatiana Ehrenfest published a very detailed article[3] on Boltzmann's statistical approach to dynamical systems with a large number of degrees of freedom. They analyzed the main objections to Boltzmann's theory, known as Zermelo's and Loschmidt's paradoxes. (See the discussion in the box on page 41). An advanced description of the paradoxes and responses to them can be found in the works of Mark Kac.[4] In fact, Zermelo's and Loschmidt's paradoxes apply not only to Boltzmann's theory, but also to any statistical theory whose results include monotonic entropy growth. Generalizing and broadening our problem, we may now ask: How can irreversible kinetic equations that appear to adequately describe physical reality follow from reversible dynamical equations? A formal resolution of the problem, following Boltzmann's idea, is based on a physically reasonable assumption known as the random phase approximation (RPA) or its equivalent,[4] which assumes rapid decay of correlations in the system.

For a fairly long time, the RPA was an effective tool in achieving a transition between the deterministic and statistical descriptions. However, further investigations were unable to justify the RPA assumption in every case, and it has remained unclear as to when and how one applies statistical (irreversible) considerations instead of dynamical (reversible) ones. One example of the ambiguity between determinism and statistics is the Fermi-Pasta-Ulam problem[5] of the emergence of thermodynamic properties in a system of coupled nonlinear oscillators. Researchers were unsuccessful in finding statistical relaxation to equilibrium when the number of oscillators becomes fairly large. It was at this point that the theory of chaotic dynamics could be applied to understand the appearance of randomness and, what is more important, the conditions needed for its appearance.[6,7]

Mixing properties in phase space are crucial for the relaxation of systems to statistical equilibrium. The notion of mixing was introduced by J. Willard Gibbs, and it is viewed by physicists as a decay of correlations of related physical variables. The decay of phase correlations means a corresponding phase randomization. To reach a statistical equilibrium in a finite time, it is important to have a finite time of mixing. Can chaotic dynamics provide this finite mixing time and thus provide the important condition of randomness that is essential for deriving any kind of kinetics? It is largely the answer to this question that is the subject of this article.

The important role of quantum effects and the connection between statistical laws and so-called quantum chaos deserves a special discussion and so are not considered here (see also reference 7). One provocative property

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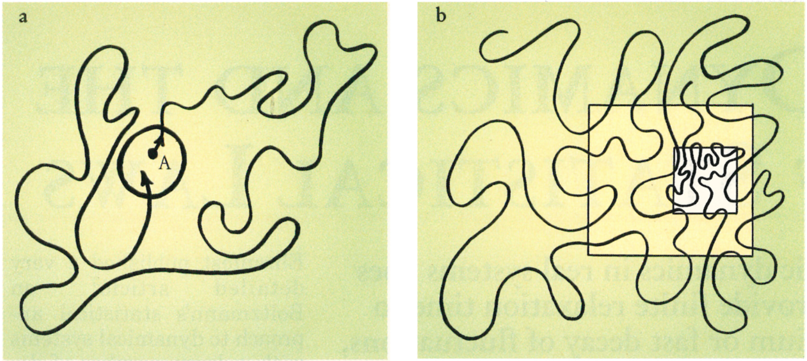

FIGURE 1. POINCARÉ RECURRENCES of a trajectory in phase space, with the trajectory (curved line) returning to a small domain A. A Poincaré cycle is the time interval between two consecutive escapes (or entrances) from (or to) the domain A. (a) Simplest case of no features in phase space and uniform mixing. (b) Trajectory that passes through singular zone (quasi-trap), denoted by the squares within squares: the smaller the square, the longer the particle stays in the square.

of quantum chaos that should be mentioned, however, is that despite the probabilistic character of quantum dynamics, quantum effects suppress the chaotic properties of classical dynamics.

Strange kinetics

Initially, it appeared that chaos theory could satisfactorily account for the conditions of applicability for randomness of phases. Chaos theory, however, could not explain how the averaging over random phases is performed to give physical observables. Fairly recent theoretical and numerical results have shifted the problem of the foundation of statistical laws from Zermelo's and Loschmidt's paradoxes to an even older one: the problem of the existence of Maxwell's demon, a device that, contrary to the second law of thermodynamics, can gather a significant part of a gas in one section of a chamber without performing any work.[8] It would seem that the new phenomenon of chaos requires the abandonment of any assumption of randomness of some dynamical variables (like phases), because, under certain conditions, the dynamical equations are, in a manner of speaking, "playing dice by themselves." From observation, however, it appears that this "dice playing" is rather strange, and the kinetics originating from chaotic dynamics is strange as well.[9] In a certain sense, chaos does not show complete randomness, and we now have to dig deeper to understand the origin of the statistical laws. Important discoveries in the search for the dynamical generation of randomness include dynamical traps and dynamical ways of cooling particles and systems.

Contrary to early expectations, the states resulting from chaos are not completely random. Even at long times, chaotic systems include elements of order, and can be distinguished from completely random systems. "Normal," or complete, randomness has exponential decay of correlations, distributions with all finite moments, and fast decay of fluctuations. The incomplete randomness of chaos, however, has a power-law decay of correlations, distributions with infinite moments, and long-lasting fluctuations. It is the incompleteness of randomness of chaotic dynamics that we describe in this article. We also discuss how this incompleteness influences the relaxation to statistical equilibrium.

The Poincaré recurrence theorem (PRT) states that in a dynamical system in which motion is confined to a finite region and phase-space volume is conserved, any trajectory returns back to any vicinity of its starting point (except for some trajectories of zero measure). Zermelo

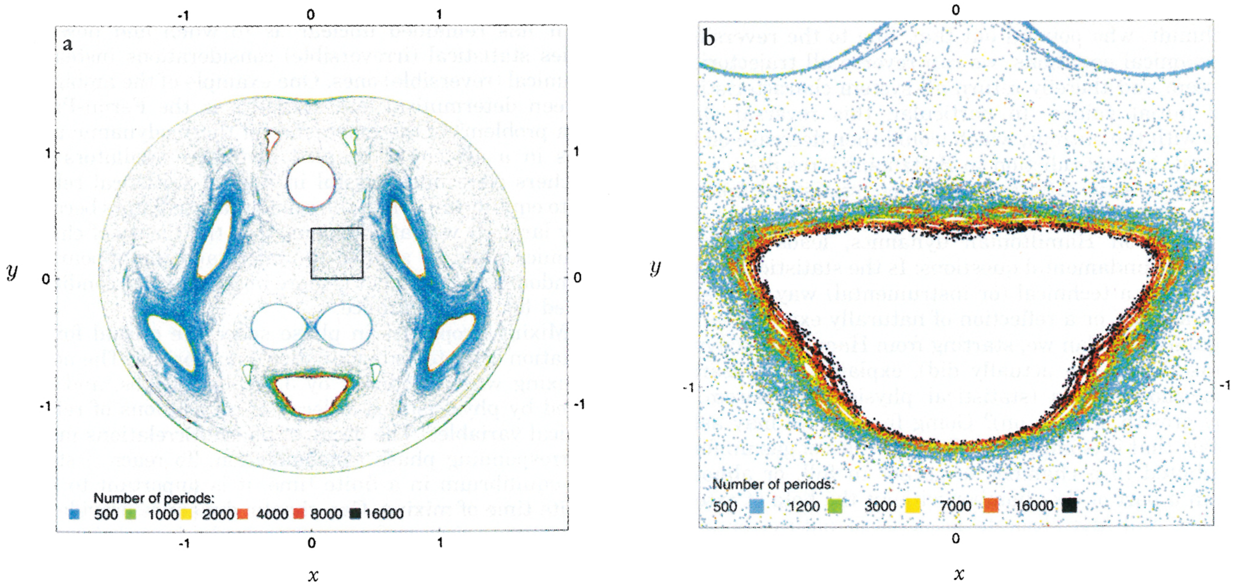

FIGURE 2. FRACTAL POINCARÉ RECURRENCES, as shown in a stroboscopic presentation of trajectories of passive particles advected within a velocity field generated by a three-point-vortices system. Each point represents a particle position in phase space at a given time (scale units are arbitrary). All particles were initially in the outlined square; colors indicate how long a trajectory needs to return to the square. Stickiness of the island boundaries is shown by the high particle density and long return time in these regions. Both the full phase space (a) and magnification of one part (b) display the complicated (fractal) structure of the sticky boundary layer, as well as the complexity of the stickiness and recurrence time.

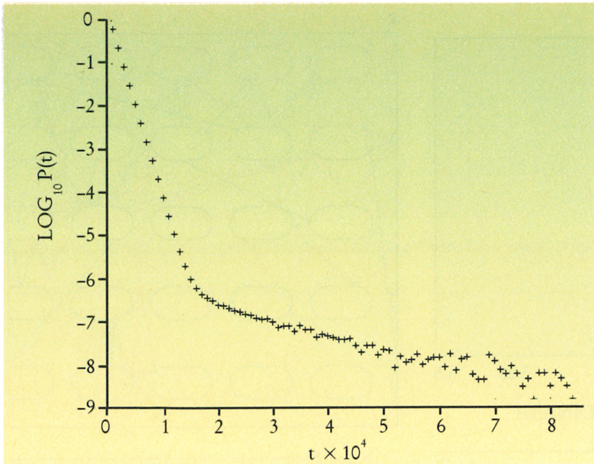

FIGURE 3. TYPICAL POINCARÉ RECURRENCE TIME distribution $P_{rec}(t)$. The distribution consists of two parts, with a Poissonian shape for fairly small time, and a power-law dependence for large time. This distribution was derived[13] for the standard map,[6] which is an iterative procedure based on the periodically kicked rotor. For more complicated systems, the late-time power-law dependence might reflect multiple exponents.

speculated that the PRT meant that the entropy of the system should follow the same property, returning close to its initial value, in contradiction to Boltzmann's statement that entropy monotonically increases.

Boltzmann's reply to Zermelo was to estimate the Poincaré recurrence time, which, for a system of 10^{23} particles, is much larger than the lifetime of the universe. Although this argument shifts the fundamental problem from quality to quantity, Boltzmann was the first to mention the importance of the mean recurrence time τ_R and to propose formulas for its estimation. He also made the important comment that the PRT does not say anything about τ_R. As it turned out, this comment concerns one of the most significant properties for the understanding of dynamical systems with chaotic motion—namely, the distribution of recurrence times.

We can define the distribution function $P_{rec}(\tau)$ of Poincaré cycles τ as follows (see figure 1a). Consider a single trajectory in phase space that passes through a finite region A and exits. According to the PRT, the trajectory will reenter the region A again and again, after spending varying amounts of time outside the region. The interval between one exit from A and the next is the Poincaré cycle time τ_R. The distribution function $P_{rec}(\tau)$ gives the probability, as the volume A goes to zero, that a Poincaré cycle will be completed in time τ. Different features of $P_{rec}(\tau)$ make it useful for studying dynamical systems. Kac[4] studied Poincaré cycles and proved an important theorem: $P_{rec}(\tau)$ exists, independent of the choice of domain A; and the mean recurrence time τ_R is finite, provided that the particle dynamics is area preserving and bounded in phase space, and that there exists a unique nonzero particle distribution function in the phase space.

Kac's result created new difficulties for the study of chaotic dynamics in various systems, including those in which the randomness (chaos) is supposed to appear even for two degrees of freedom, those in which the existence of a positive measure of chaotic orbits has not yet been proved even for simple cases, and those in which nothing is known about the uniqueness of the distribution function.

The distribution of Poincaré cycles $P_{rec}(\tau)$ possesses the unique property that it characterizes the full phase space, including possible nonuniformities and singularities. We can say that obtaining the Poincaré recurrence distribution is a way to visualize the chaotic dynamics. If, for example, there exists a singular domain A_s (discussed below), then some of the cycles passing through A_s carry information about that singular domain (see figure 1b). Furthermore, it is possible to identify different types of singular domains in phase space. Some of the domains are called quasi-traps or dynamical traps, and they can be recognized by the distribution of recurrence cycles $P_{rec}(\tau)$.

Flights and dynamical traps

A typical phase space of Hamiltonian dynamics looks bizarre and resembles a topological zoo, consisting of domains of chaotic dynamics with such elements as the stochastic sea, stochastic layers, and stochastic webs, as

Entropy paradoxes and chaotic dynamics

Just after Ludwig Boltzmann's publication of his H-theorem on the monotonic increase in entropy until the system reaches its equilibrium, a sharp discussion around his theory was initiated by Ernst Zermelo and Josef Loschmidt, who formulated their objections in the form of paradoxes.

Zermelo's paradox of recurrence: According to the Poincaré recurrence theorem, any state of the system will be repeated within any prescribed accuracy infinitely many times. Zermelo stated that entropy should be repeated as well, in contradiction to the Boltzmann result.

Loschmidt's Paradox of Reversibility: The equations of motion in mechanics are time reversible. Therefore, in addition to the process that leads to increasing the entropy, there should be a backward process that leads to decreasing the entropy.

Two important features of statistical systems should be involved in resolving the paradoxes: the fantastically large number of particles in a system (about 10^{23}) and the coarse-graining procedure, as carefully analyzed by Paul and Tatiana Ehrenfest, that unambiguously leads to the neglect of immensely small probabilities for recurrences and of reverse, entropy-decreasing processes.

Chaotic dynamics provides a new concept of mixing in phase space and a new understanding of the two paradoxes. First, we note that the Poincaré recurrence theorem has nothing to do with the appearance of statistical properties in a system. Recurrences exist in both quasiperiodic and stochastic motions. For a small number of particles (even two!), chaotic dynamics leads to progressive increases in the complexity of the shape of an initial phase-space droplet; for the evolved droplet to return to its initial state would be an event of incredibly small probability. "Coarse graining" means that a state is defined up to a region of small volume Δ. After a while, the initial droplet of the phase space is well mixed over the finite phase space, so that the domain Δ consists of trajectories that could be initially at any available region of the same volume in the full phase space. Which trajectory should be taken to perform Loschmidt's backward dynamics? Any information about the initial states of trajectories disappears after the coarse graining (see more discussions in reference 6).

In this article, we indicate that chaotic dynamics *per se*, despite its success in resolving the two paradoxes described here, has the possibility of long-lasting fluctuations (such as bursts, flights, and traps), which prevent the system from obeying the standard laws of thermodynamics for an arbitrarily long time, providing that no additional procedure or assumption has been made.

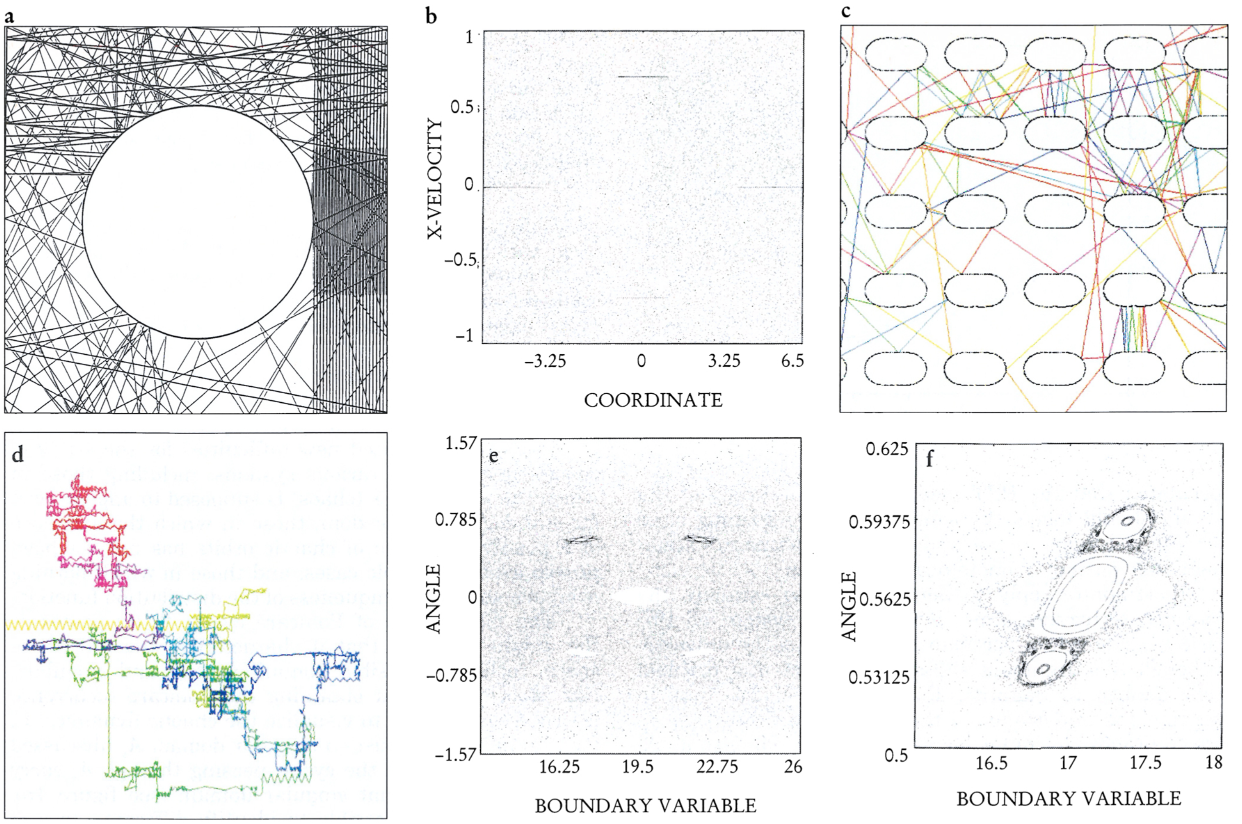

FIGURE 4. POINT PARTICLES, BOUNCING OFF BILLIARDS of various shapes, generate two-dimensional trajectories with long flights, stickiness, and singular zones. (a) Sinai (circular) billiard gives rise to a long bouncing trajectory. (b) The phase-space mapping of the trajectory in (a) displays scars (the four horizontal lines), which are domains of zero phase volume where trajectories cannot enter. (c) Cassini (oval) billiards in a periodic array generate trajectories in a system similar to the Lorentz gas. (d) The same trajectory as in (c), viewed at larger scale, displays long flights and trappings. (e) The phase-space mapping of the same trajectory displays both island structure and stickiness. (f) Magnification of an island in (e) reveals islands surrounding the island. The special parameter values $a = 4.0309525$, $c = 3$ for the shape of the Cassini oval result in satellite islands being generated at each successive magnification; with a proliferation number of 4-8-4-8. . . , denoting the formation of first 4, then 8, then 4 islands, and so on.[15]

well as islands filled by periodic and quasiperiodic orbits and smaller domains of chaos. The periodic and quasiperiodic orbits inside the islands are called, using a standard terminology, KAM (Kolmogorov-Arnold-Moser) invariant curves. These orbits are stable, and their presence, or the presence of islands of finite phase volume, makes the dynamics nonergodic. When chaos was first studied, it seemed that the existence of the islands was not very important for determining the origin and character of randomness. One reason was that the volume of the islands can be very small; another was that the phase space of islands can be excluded from consideration, after which the rest of the phase volume, called the stochastic sea, becomes ergodic. Subsequently, however, numerous investigations overturned that optimistic hope, shifting the focus of interest from the domains of KAM orbits to the vicinity of boundaries of those domains, which have much smaller phase volumes. Crossing an island boundary, we jump from a regular (periodic) orbit to the chaotic one that lies in the stochastic sea. The vicinity of the island boundary is *terra incognita*, and, despite significant mathematical effort,[10] it is still poorly understood how a trajectory shifts from regularity to the chaotic regime.

Simulation shows that the vicinity of an island, called the boundary layer, is sticky. This means that a trajectory spends more time in the boundary layer than in a domain of the stochastic sea of the same phase volume but located far from the island (figure 2). The island's boundary can be more or less sticky, depending on the control parameters of the system. There are also special zones located near the island's boundary where a trajectory can be trapped for a long finite time, and the size of these zones depends on the system parameters. The phenomenon of trapping becomes crucial for our understanding of Hamiltonian chaos, and it can play an important role in various applications. Below are few examples of the trapping phenomenon:

a. Chaotic advection. This example describes a so-called Lagrangian (or passive) particle dynamics in a fluid flow $\mathbf{v}(\mathbf{r})$

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r}). \tag{1}$$

For incompressible flow with div $\mathbf{v} = 0$, equation 1 corresponds to Hamiltonian dynamics. In figure 2, we present a stroboscobic plot of a particle trajectory when the velocity field $\mathbf{v}$ is generated by three interacting point vortices.[11] This problem has numerous applications in geophysics, in which the chaotic motion of particles (tracers) in a given velocity field is known as Lagrangian turbulence. A trajectory started in the outlined square of figure 2a returns back to the square repeatedly with different time intervals that are marked in colors. Long returns correspond to parts of trajectories that stick to the boundary of islands or subislands and so on. The existence of the sticky islands and their hierarchies results in anomalous diffusion in

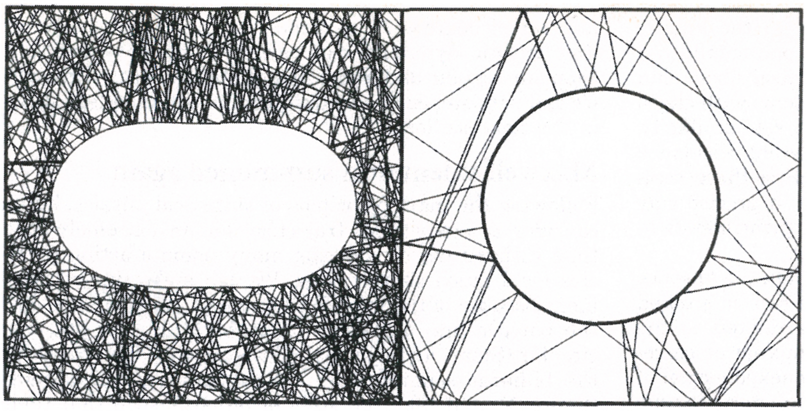

FIGURE 5. MAXWELL'S DEMON AT WORK in a system with two different scatterers in a box, separated by a wall with a small window in the center. The scatterers are billiards, with the left Cassini (oval) and the right Sinai (circular), but the phase volumes on each side are equal. We launch test particles to scatter off the billiards, and measure the distribution of time intervals that a particle spends in each half-box before escaping through the window. Surprisingly, for certain parameters, relaxation of the system does not occur, even for times immensely greater than the mixing time. Differences in the mean recurrence times can be interpreted as the difference in effective pressures in the left and right boxes.

azimuthal angle θ (an angle of rotation about some central point of the domain)

$$\langle \Delta\theta^2 \rangle = \langle (\theta - \langle\theta\rangle)^2 \rangle \sim t^{\mu} \tag{2}$$

with transport exponent μ, which depends on a control parameter (geometry of the vortices) in a nonsmooth way. For the case in figure 2, $\mu \sim 1.6 > 1$, corresponding to superdiffusion. Different instances and mechanisms of superdiffusion have been observed in many physical situations, including passive particle motion in Beltrami-type flow, turbulent diffusion, charged particle dynamics, and advection in a rotational tank.[12]

A typical distribution of Poincaré recurrences $P_{\rm rec}(t)$ is shown in figure 3 for a well-known chaotic system, the standard map (corresponding to a periodically kicked rotor). The distribution follows the Poissonian law up to some crossover time t^*, after which it has a power-law behavior

$$P_{\rm rec}(t) \sim t^{-\gamma}. \tag{3}$$

There is a connection between γ and μ, but it depends on many factors. For the problem in figure 3, renormalization group theory was used to show that $\gamma = \mu + 2$.[13]

The power law (equation 3) occurs as a result of long trappings (or long flights, as in the next example) in the phase space. We can define a trapping time distribution $\psi(t,\Delta\Gamma_A)$ based on the amount of time that a trajectory spends in the domain $\Delta\Gamma_A$ on each recurrence. For any domain inside the trapping zone, $\psi(t) \sim P_{\rm rec}(t)$ when $t \gg t^*$.

More generally, trapping domains correspond to a type of singular zone in phase space. These zones can be characterized either by the distribution $\psi(t,\Delta\Gamma_A)$, which depends on the location of the zone $\Delta\Gamma_A$, or by $P_{\rm rec}(t)$, which does not depend on locations of different zones and which represents a cumulative characteristic of the phase space. In general, the set of recurrence cycles $\{\tau_j\}$ cannot be characterized by only one exponent γ, and it is necessary to introduce some distribution of different values of γ over the range of $P_{\rm rec}(t)$.[13] The use of one value of γ is a rough approximation good for some special values of control parameters and time windows.

b. Billiards. Billiards is a very easily visualized example of the existence of trapping domains. Here, we consider a two-dimensional problem of a point particle elastically scattering from a billiard ball enclosed in a box with perfectly reflecting walls. Two kinds of billiards are presented in figure 4: the Sinai billiard with a circular scatterer, and the Cassini billiard with an oval scatterer given by the curve $(x^2 + y^2)^2 - 2c^2(x^2 - y^2) - (a^4 - c^4) = 0$.

Their phase spaces differ. For the Sinai billiard, there are no islands but there are "scars"—nonreachable domains of the phase space of zero measure (figure 4b). Any trajectory has parts that correspond to arbitrarily long bounces, or flights in the space (for example, the long trajectory on the right of figure 4a).

The Cassini billiard (figure 4c) shows similar flights and trappings (figure 4d), but there are also flights due to the presence of islands in the phase space. One can find an example of parameters a, c for which there exists an infinite hierarchy of islands with especially strong stickiness. Figure 4e shows the phase space for such parameters with an island in the center, surrounded by four other islands. Magnifying one of these islands, as in figure 4f, it can be seen that each surrounding island is surrounded by four others. Further magnification shows that each of the islands is surrounded by eight islands, each of which is surrounded by four islands, and so on. This infinite hierachy is labeled 4-8-4-8 . . . to denote the number of surrounding islands at each level of magnification.

The recurrence time distributions for billiards exhibit the same kind of behavior as in figure 3: $P_{\rm rec}(t)$ is Poissonian in form below a certain crossover time, above which it transitions to a power-law tail with $\gamma \approx 3$ for the Sinai billiard[14] and $\gamma \approx 3.15$ for the Cassini billiard.

Dynamical traps vs. chaos

The presence and variety of singular zones makes the dynamics of each chaotic system individual in some sense. The bad news of this loss of universality is offset by the good news that the dynamics in a singular zone determines the large-timescale behavior of systems. As an example, we can consider a hierarchical dynamical trap (figure 1b) consisting of an infinite set of nested domains with phase space volumes $\Gamma_0 > \Gamma_1 > \ldots$.[15] A typical trajectory fills the phase space almost uniformly because of the mixing property of chaos, except for a small part $\Gamma_1 < \Gamma_0$ where the trajectory stays time T_1. Time T_1 is longer than the time T_0 that the trajectory spends outside of Γ_1. We can increase the time of consideration and resolution to observe in greater detail the behavior of the trajectory in Γ_1, find a hierarchical similarity, and so on. We say that there exists a self-similar trap if

$$\Gamma_n = (\lambda_\Gamma)^n \Gamma_0\,, \quad T_n = (\lambda_T)^n T_0\,, \quad (\lambda_\Gamma < 1,\ \lambda_T > 1), \tag{4}$$

with appropriate scaling parameters λ_Γ and λ_T.

Actually, hierarchical traps have been observed in many different models[13,15] for special control parameter values (the trapping conditions of systems strongly depend on the control parameter). Usually, the self-similarity condition is more complicated than equation 4, and there are different traps with different scaling parameters λ_T, λ_Γ distributed in some interval of values. Traps correspond to a very complicated spatial–temporal coherent structure in the phase-space dynamics, with strong and far-lasting fluctuations that should be accounted for in the kinetic description of chaotic systems. Of course, one should not conclude that the simplified version described above can be typically observed in real dynamical systems with chaos since real systems have noise, lack exact self-similarity, have different degrees of freedom with different resonance properties, and so on. But before confronting these real-world obstacles, we need to understand in a precise way what can or cannot be derived from first principles (say, from the Hamiltonian equation of motion).

The presence of traps is more typical than the absence of traps for Hamiltonian dynamics. Traps can be studied in at least three ways. The first way is related to the understanding of chaotic dynamics: The mixing or correlation decay property of chaos can be nonexponential if the timescale of consideration exceeds a characteristic time of trapping. For long timescale, a trap can be considered as a scene in which a process with fractal time occurs.[16] The second way is to consider transport properties of particles or some macroscopic moments of distribution functions. In the presence of traps, the transport is anomalous (non-Gaussian). In fact, the transport exponent μ in equation 2 can sometimes be related to λ_Γ and λ_T. For example, for some special cases,[13]

$$\mu = | \ln \lambda_\Gamma | / \ln \lambda_T , \qquad (5)$$

which represents a coupling between space and time scaling and the transport exponent. The third way is related to the foundation of statistical physics, and is discussed in the following sections.

Loss of universality of chaotic dynamics

The existence of different traps leads to a kind of nonuniversality of distribution functions in chaotic dynamics that is contrary to what we are accustomed to in thermodynamics with Gibbs distributions. In statistical mechanics, there typically exists the thermodynamic limit, which provides an equilibrium distribution, in analogy to the large number theorem. As was shown by Paul Lévy, the Gaussian distribution is not the only one with the same form on both large and small scales[17] (see the article on Lévy flights by Joseph Klafter, Michael Schlesinger and Gert Zumofen, PHYSICS TODAY, February, 1996, page 33). The Lévy distribution $p_n(x)$, like the Gaussian, has the property that the sum of any number of functions $p_1(x)$ has the same form as $p_n(x)$. The Lévy distribution includes a parameter $0 < \alpha < 2$ for which it is positive definite, and it is identical to the Gaussian distribution when $\alpha = 2$. The parameter α is related to the fractal dimension of the space of random events, or flights.[16]

Processes induced by Hamiltonian chaotic dynamics are much more complicated than the random walk considered by Lévy, and we can assume on the basis of simulations that there can be classes of universality rather than universality. The sources of nonuniversality will be the different nontrivial elements of phase-space structure (like islands of different orders of resonances, separatrices, and boundaries). Because of these structures, fluctuations from a stationary state can be arbitrarily large, and have a nonsmall probability of occurrence (compared to Gaussian fluctuations, which decay exponentially). Moreover, moments of the fluctuations higher than second order diverge. An intrinsic property of chaotic dynamics is that for a given arbitrarily large dimensionless time t^* (for example, 10^{10}), we can find ranges of the system control parameters for which relaxation to an equilibrium or stationary state will not happen with a finite (nonexponentially small) probability. Of course, this property can be established now only for a system with few degrees of freedom—nothing is known about cases with a large number of degrees of freedom. Nevertheless, we can see that chaotic dynamics can possess a property different from our regular understanding of randomness, a property we call persistence of nonequilibrium. An example is given in the next section.

Maxwell's demon is summoned again

Following the main principles of statistical physics, let us consider a one-particle trajectory for an extremely long time rather than considering many noninteracting particles for a much shorter time. We can make this replacement because of the ergodic property of the dynamics. Put the particle into a system of two billiards in a box, one circular (Sinai) and one oval (Cassini), with the spaces for the billiards separated by a wall with a small window (figure 5), and take the sizes of billiards such that their phase volumes are equal (not an easy task, because of the infinite fractal sets of islands in the Cassini billiard part). Then define the residence time to be the time that a particle spends in either the Sinai or Cassini half-box, between entering and exiting the half-box. There are a few questions that cannot be answered trivially: What will be the distribution functions of particle residence time $P_S(t)$ and $P_C(t)$ in the Sinai and Cassini half-boxes? What are the moments of $P_S(t)$ and $P_C(t)$ and, particularly, what are the mean residence times? In fact, the distributions $P_S(t)$ and $P_C(t)$ are none other than the distributions of the Poincaré recurrences to a domain near the window of contact from the right and left side, respectively.

To increase the effect, we can adjust the parameters (a, c) of the Cassini oval to have the self-similar hierarchy of islands discussed earlier, and hence the strongest stickiness (see figure 4), while balancing the phase volumes of the Sinai and Cassini parts. The results are astonishing: Only for times less than a crossover time t^* are the distributions of Poincaré recurrences for both sides identical and Poissonian; for $t > t^*$, the distributions are visibly different. This difference persists for a computational time $t_{\max} \sim 10^{10}$ cycles (figure 6), incomparably greater than the mixing time, which is of the order 10. This result can be viewed as a consequence of an action by an invisible Maxwell's demon whose role is played by a distributed specific topological structure of the phase space created by a special form of the Cassini oval.[15] (In our model, the demon is not localized as it was in Maxwell's original definition, but its essential role is the same, as a device embedded in the system that can strongly modify the system's thermodynamic properties). Long-lasting fluctuations prevent the relaxation in a finite time. As a result, the relative differences between the mean recurrence times τ_C and τ_S—the first moments of $P_S(t)$ and $P_C(t)$—can be interpreted as the difference in effective pressures in the left and right boxes. The difference increases if we consider the higher-order moments, which are finite for a finite observation time but grow to infinity (!) because of the power-wise tails of the distributions $P_{C,S}(\tau)$.

The above demonstration raises a new question: What kind of thermodynamics should describe systems, like billiards, with islands in their phase spaces? This question is relevant to typical Hamiltonian systems, especially if we take into account that a smoothing or softening of billiard borders generally creates an island structure in the phase space. The example with billiards can be extended to real systems like the advection with point vortices discussed above, in which two different sticky zones can play the role of the two chambers. Considering a contact between the zones for different values of the control parameter, we could expect to find no typical thermodynamic equilibrium for the gas of advected particles during an astronomical time.

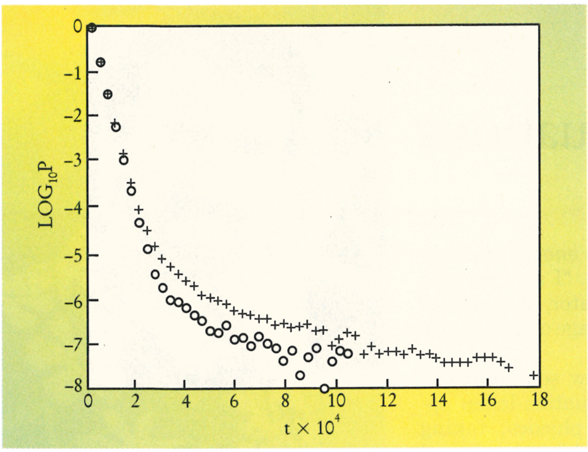

FIGURE 6. FOOTPRINTS OF MAXWELL'S DEMON. The Poincaré recurrence time distribution $P_{rec}(t)$ for the system depicted in figure 5 shows no evidence of relaxation for $t > t^* \sim 2 \times 10^4$, where t^* is the crossover time. When the two systems interact for many cycles, this plot shows the recurrence time distributions in the left (crosses) and right (circles) half-box. No relaxation to identical distributions is seen, even after the computational time 1.16×10^{10} for 37 trajectories.[15]

When is chaos random?

There is a kind of heresy in this question. Long ago, in an attempt to formalize our inability to predict the evolution of some processes, the notion of randomness was introduced and random processes were mathematically invented. One could recall the axiomatic way in which randomness was introduced into our toolbox and justify its usefulness by citing an incredible number of successful applications. Contemporary scientific achievements have expanded our possibilities in such a way that we can observe in reality a new phenomenon called chaotic dynamics. This phenomenon is generated (or described) by nonrandom, reversible, regular equations. At the same time, the same equations describe dynamics that, in some sense, lies between regularity and randomness. We need a realistic process that possesses properties of both regularity and randomness in different proportions, so that the combination is not just a plain mixture of both kinds of properties. Today's computational power and instrumental analysis make it possible to distinguish chaos from randomness, and even control and erase chaos or make predictions from it. At the same time, we must look for ways to describe a chaotic system complete with traps, flights, power-type distributions, infinite moments, and so on.

To put it simply, in dealing with chaos, we should be prepared to accept a kind of thermodynamics without a monotonic evolution of the system, a stationary statistical distribution without a finite time of relaxation of fluctuations, and kinetics with a partly (at least for finite time) predictable evolution.

Our current knowledge leads to a view that randomness in its original, axiomatic definition is rather an approximation to chaotic orbits that are solutions of pure dynamical Newtonian, Maxwellian, or Einstein equations. This approximation, whether good or poor, does not guarantee the occurrence of the traditional statistical physics condition of a finite time relaxation to the equilibrium state and fast decay of fluctuations. We have to think again about the derivation of precise criteria for the occurrence of statistical laws to replace the so-called thermodynamic limit (N, $V \to \infty$; $N/V =$ const), which is more a way to obscure the situation than to solve the problem of the foundation of statistical physics and the origin of statistical laws.

Other implications

Our discussion of the origin of statistical laws would be deficient without mentioning some direct applications of the incomplete randomness of chaos and the persistence of nonequilibrium. One such application is a stationary thermonuclear reactor, a device containing collisionless chaotic dynamics and strongly intermittent processes, especially in the plasma edge zone. Although the number of particles N is normally very large and no one has doubts that the volume V is also sufficiently large, the "thermodynamics" of the operating reactor or its edge part is rather far from a usual meaning of this notion. Although little can be said about the thermodynamics of fusion, two other directions are quite rich in observations: the transition state to developed turbulence and the large fluctuations of localization length seen in quantum chaos.[18] These subjects deserve to be illuminated separately and in more detail.

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