

SEARCH AND DISCOVERY

Small-World Networks Can Provide a New Tool to Study Diverse Systems

The concept of six degrees of separation, as studied over three decades ago by the social psychologist Stanley Milgram, grew from an experiment in which two target persons were identified in the Boston area, and then individuals in Kansas and Nebraska were each sent a letter describing some parameters of the two targets. Each individual was given a folder with the rule for reaching the target: If the individual didn't know the target, he was asked to send the folder to a friend he knew on a first-name basis who was likely to know the target person. Milgram's surprising result was that the number of links needed to reach the target person had a median value of six.¹ This concept is also known as the small-world phenomenon.

Recently, Cornell University mathematicians Duncan Watts and Steven Strogatz have found that the small-world phenomenon is a general property of large, sparse networks as disparate as the so-called collaboration graph of film actors, the neural network of the worm *Caenorhabditis elegans* and the power grid of the western US.² Two extreme kinds of topology have been used to model networks. In many physical systems—such as Josephson junction arrays, laser arrays or coupled oscillators—the connection topology is a regular lattice. For models of genetic control networks or disease spreading, the connections are often assumed to be random. But many other networks, says Watts, appear to lie somewhere in between these two extremes. In their paper, Watts and Strogatz describe a simple model they developed for interpolating between a completely ordered network and a random one, by means of a “random rewiring” process.

Finding shortcuts

In figure 1, a simplified version of the mathematical model studied by the Cornell mathematicians, they start from a ring lattice with n points (or vertices), in this case, $n=20$. Each point is linked to k other points;

▶ If you are well connected and take a few shortcuts, you might be able to model systems in physics, biology, chemistry or computer science.

in this case, the number of links (or edges or bonds), k , is 4. They then rewire each link at random with probability p . This construction lets them “tune” the graph between complete order ($p=0$) and complete disorder ($p=1$) and thus probe the intermediate region.

They choose a point and the links connecting it to its nearest neighbor moving in a clockwise direction. To randomly rewire, they could pick a new vertex from anywhere in the lattice and connect the original point to it.

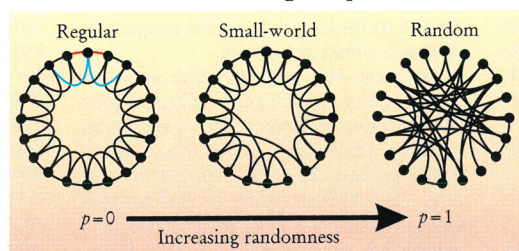


FIGURE 1. RANDOM REWIRING PROCEDURE for tuning the graph of a ring lattice (left) between order, $p=0$, and disorder, $p=1$ (right), allows the small-world region (middle) to be probed. The topmost point in the ring lattice has four links—two red arcs and two blue parabolas. The small-world network is highly clustered like a regular graph, but it has a small characteristic path length like a random graph (see figure 2). (Adapted from ref. 2.)

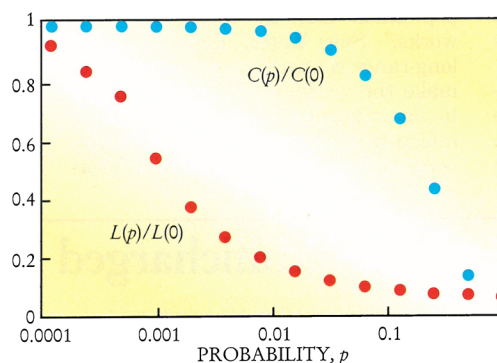


FIGURE 2. CHARACTERISTIC PATH LENGTH $L(p)$ and clustering coefficient $C(p)$ for the family of rewired graphs in figure 1. The rapid drop in $L(p)$ is driven by the existence of a small fraction of long-range links that connect previously distant parts of the network. (Adapted from ref. 2.)

Such a random connection can give you a long-range connection. Then the researchers repeat the process by moving clockwise around the ring, considering each link in turn (with probability p) until every link has been considered once. Figure 1 shows the original ring, an intermediate (small-world state) and a completely random state.

Watts and Strogatz define a characteristic path length $L(p)$, which represents the average number of steps (or degrees of separation) in the shortest path between randomly chosen points in a network.

They also define a clustering coefficient, $C(p)$, which measures the average local density of the connections. For friendship networks, L is the average number of friendships in the shortest chain connecting two people; C reflects the extent to which friends of a given individual are also friends of each other, and corresponds to the typical cliquishness of each individual's personal network of friends.

Figure 2 shows $L(p)$ and $C(p)$ for a network of 1000 points with $k=10$, averaged over 20 random realizations of the construction algorithm in Figure 1 and normalized by their respective values at $p=0$ (the ring lattice). The length and clustering statistics turn out to behave quite differently as a function of increasing randomness—for very small p , as p increases L drops rapidly, but C remains virtually unchanged. The resulting class of networks that exhibit small L and large C are what Watts and Strogatz call small-world networks. Watts explains, “The rapid drop in $L(p)$ is driven by the presence of a small fraction of long-range links that connect previously distant parts of the network, bringing together not only the two points involved, but also their neighbors, their neighbors' neighbors and so on.”

“This result is actually quite general,” says Watts (who will shortly be moving to the Santa Fe Institute in Santa Fe, New Mexico), “and does not depend on the choice of a ring substrate

used in the model. All that is required to generate the small-world phenomenon is a network that is locally ordered (which means simply that two nodes with a mutual 'friend' are significantly more likely to be connected than two randomly selected nodes), and which has a small fraction of long-range shortcuts. The effect also does not depend on the specific nature of the network nodes or connections—only their topology—so the small-world phenomenon ought to arise in all sorts of large, sparse networks.”

To check this, Watts and Strogatz examine the length and clustering properties of three real networks: the collaboration graph of movie actors (including approximately 225 000 actors of all nationalities since the start of motion pictures); the power-transmission grid of the western US; and the neural network of the worm *Caenorhabditis elegans* (the only organism whose neural network is completely known). As Watts explains, they show that, in each case, the characteristic path length of the network is close to its theoretical minimum (that of an equivalent random graph), yet the clustering coefficient is far from minimal, indicating the presence of significant local order. So all three networks exhibit the small-world phenomenon.

Assuming that the appropriate data can be compiled, Watts thinks science citations would be an interesting network to study, with each point being a scientific paper and a directed link signifying a citation. A related example is the collaboration graph of scientists, traditionally centered around the late mathematician Paul Erdos.

“You could apply small-world network ideas to words, which can be connected because they are synonyms or antonyms or have similar roots,” says Watts. “The structure of this network may then be relevant to the flow of ideas in a conversation or the cognitive paths leading to creative insight.”

The Internet is an interesting network, of course. Says Watts, “Imagine a vertex being the Web page and the hot links being the connections. There

are probably millions of pages on the World Wide Web, and very few of these would have more than 100 hot links, yet it seems possible that any Web page could be reached from any other in only, say on the order of 10 links.”

Dynamical properties

Watts first started thinking about the small-world phenomenon in connection with coupled dynamical systems. While a graduate student at Cornell, he was studying the synchronization of populations of coupled oscillators, in particular crickets. He wondered if for a fixed population of oscillators and a fixed number of connections between them, the dynamics could be affected by altering the distribution of connections. More specifically, Watts asked, if certain coupling networks could be shown to have the property that all elements were, in a sense, close to all others, would those networks be predisposed to synchronize more easily or rapidly than, say, one-dimensional lattices?

Watts and Strogatz, having shown that small-world networks could arise under rather general conditions, then turned to finding dynamical properties of small-world networks.

Suppose the points of the network are neurons or a person telling a joke or spreading a disease, says Strogatz. In their simplified model the first infected person remains infected for one time step during which he has a probability, r , of infecting each of his healthy neighbors, and so on. If r isn't large, it's possible the disease would die out. The team finds that diseases “spread like wildfire through a small-world network. It takes very few shortcuts to make the time for global infection nearly as short as a random graph,” according to Strogatz. In a small-world network, diseases spread rapidly and tend to infect a larger fraction of the population than in “large-world networks.” Says Watts, “Because so few long-range connections are required to make the world small, it may appear to an individual that the danger of infection is remote whereas it in fact is quite close.” William Ditto at Geor-

gia Tech is now doing experiments to see what role small-world networks might play in epileptic seizures.

Another dynamical system that Watts and Strogatz have studied is a binary-state cellular automaton, in which cells are either on or off. The computational task of the automaton is to say whether the fraction of “on” cells is greater or less than one-half. James Crutchfield and Melanie Mitchell (Santa Fe Institute) called this task “density classification.”³ If the fraction is greater than one-half, the cellular automaton has to turn all cells on. If it's less, then it has to turn them all off. Although this sounds like a trivial problem, it's difficult for a locally connected system. And it's representative of a class of other problems, such as the functioning of an ant colony, which through local interactions alone appears to be coordinated globally.

Crutchfield and Mitchell used a genetic algorithm (which mimics Darwin's rule of natural selection, where the automata reproduce more automata in proportion to their performance at the computational task). Watts and Strogatz have tried a different approach, altering the architecture of the network using the model shown in Figure 1. They find that small-world cellular automata can outperform even the best performance of the genetic algorithm approach by using a naive “majority rules” approach, in which each cell simply adopts the state exhibited by the majority of its neighbors.

Says Strogatz, to do parallel computing, optimizing the architecture is important. What is perhaps less obvious, he says, is that the new architecture can be close to the original, but you don't need much randomness to improve performance.

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References

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Search at SLAC Finds No Millicharged Particles

The quantization of electric charge is very well tested: The proton and electron charges $\pm e$ are equal and opposite to better than a part in 10^{21} , and the neutrality of the neutron, photon and neutrino have been experimentally affirmed with comparable precision. And all attempts to find naked fractionally charged quarks have failed thus far, as the theory of quark con-

No established theory excludes the possibility of particles with charges a thousand times smaller than the electron's. So why not look for them, especially when you already have just the right accelerator?

finement tells us they must.

So why would a group at the Stan-

ford Linear Accelerator Center spend several years looking for elementary particles with electric charge on the order of $10^{-3}e$? The SLAC group, headed by John Jaros, has described the final outcome of its search in the 10 August *Physical Review Letters*.¹ If they had found any such “millicharged particles,” dear reader, you would already have heard about it.