Nonlinear Physics . . .

ONE OF THE MOST widely used concepts of physics is that of linearity or superposition. From a mathematical point of view, a linear system is described by equations—algebraic, differential, or integral—that are linear in the dependent variables. From a physical point of view, a system is linear or possesses a superposition principle if various modes of behavior of the system that arise from various causes or initial conditions can be added together algebraically to produce the same behavior that would result if the causes or initial conditions were added algebraically and applied to the system. For example, water waves of small amplitude superpose linearly on the surface of a pond, so that the ripples produced by two disturbances at the same time are just the sum of those produced by the separate disturbances. In most cases, the assumption of linearity is believed to be justified either as an exact representation of the situation or as a close approximation to the truth. Thus the water waves referred to above form a linear system only so long as the amplitude is small; otherwise, the shape of the resultant wave is different from that of the component waves, as is apparent when the wave is about to "break". There are some systems, however, that are not even approximately linear, and yet are commonly described by linear equations for want of more effective mathematical methods.

The assumption of linearity, whether justified or not, is an attractive one to make. For one thing, physical insight into a system can be acquired with some readiness if one knows that complicated types of behavior are describable as superpositions of simple types that can be studied one at a time. For another, mathematical techniques for handling linear equations are quite completely worked out, and are for the most part known to physicists. In contrast, even a relatively simple nonlinear system presents few points at which the physicist can come to grips with his insight, and these generally occur where the system is approximately linear. This difficulty probably stems from the fact that little mathematical progress has been made with nonlinear equations; what few methods have been developed are not widely enough known to physicists in useful form. Analytical investigations concentrate for the most part on nonlinear oscillators with one or two degrees of freedom, and even then the most generally applicable results deal either with the steady state or with perturbations on appropriate linear systems. Numerical methods can in principle solve any set of nonlinear equations, but because of the lack of superposition, many solutions must usually be obtained before the main characteristics of the behavior of the system can be inferred. This means that a large amount of computational time and effort must be expended, or that one of the recently developed high-speed calculating machines must be used. Indeed, it now seems quite probable that the quickest way of acquiring insight into the behavior of all but the simplest nonlinear systems will prove to consist of the systematic examination of a large number of numerical solutions obtained on such machines.

In some areas of physics where the assumption of linearity is successfully made, further examination shows that conditions can be found, occasionally with disconcerting ease, under which nonlinearities cannot be ignored. For example, hydrodynamics is often presented as a linear theory, and as such describes quite accurately such diverse phenomena as the increase in inertia of a solid moving through an incompressible fluid, and sound waves in gas. However, the resistance encountered by the solid cannot be calculated without taking account of turbulence, and it is now generally agreed that this cannot be done correctly without including the nonlinear terms in the Navier-Stokes equation. These terms, along with the nonlinear elasticity of gas, are also essential to a theory of large amplitude sound waves, or of shock phenomena.

Quantum field theory presents an interesting dualism in this respect. On the one hand, all of quantum theory, current field theory as well as the Schrödinger equation, is strictly linear in the sense that all observable physical quantities are represented by linear operators that act on superposable wave functions or state functionals that describe the physical system. This does not, of course, imply that the ultimate theory (if such ever exists) will have this characteristic; but thus far, no detailed consideration has been made of any quantum theory which is not linear in this sense. On the other hand, the equations of motion of the operators that represent the field amplitudes in quantum field theory can (but need not) be linear only so long as each field is considered by itself: electromagnetic field, Dirac electron field, neutron or proton field, meson field. As soon as two or more of these fields interact, the equations of motion become nonlinear, and the difficulties encountered elsewhere in physics with nonlinear systems reappear, if anything complicated by the nondenumerably infinite number of degrees of freedom that characterizes a field in continuous space-time. As elsewhere, principal reliance is placed on perturbation approximations; these are very useful where the interaction is weak, as in the electromagnetic-electron case, but of doubtful value where the interaction is strong, as in the neutron-proton-meson case.

Partly because many of the simpler problems have been solved, and partly because improvements in experimental methods often point up the inadequacies of the simpler theories, more and more attention is being paid to nonlinearities where they are known to occur, and to the possibility of their existence where they had not previously been recognized. The study of nonlinear physics has proved rewarding to the hardy in the past, and seems likely to lead to important discoveries in the future as physical insight, analytical techniques, and numerical methods progress together.

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