The Physics of Granular Materials

The rich dynamics of these ubiquitous and important materials are just beginning to be understood. Now there are suggestions that processes taking place on astrophysical scales may mirror those occurring in a pile of sand.

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Who could ever calculate the path of a molecule? How do we know that the creations of worlds are not determined by falling grains of sand? —V. Hugo, *Les Misérables*

Victor Hugo suggested the possibility that patterns created by the movement of grains of sand are in no small part responsible for the shape and feel of the natural world we live in. Certainly, granular materials, of which sand is but one example, are ubiquitous in our daily lives. They play an important role in industries, such as mining, agriculture and construction. They also are important in geological processes, such as landslides and erosion and, on a larger scale, plate tectonics, which determine much of Earth's morphology. Practically everything we eat started out in a granular form and the clutter on our desks is often so close to the angle of repose that a chance perturbation can create an avalanche onto the floor.

Hugo hinted at the extreme sensitivity of the macroscopic world to the precise motion or packing of the individual grains. We may nevertheless think that Hugo overstepped the bounds of common sense in relating the creation of worlds to the movement of simple grains of sand. This article is intended to show there is sufficient richness and complexity to granular motion that the possibility of Hugo's metaphor having literal meaning might no longer appear so far-fetched: What happens in a pile of sand on a tabletop is relevant to processes taking place on astrophysical scales.

Granular materials are simple. They are large conglomerations of discrete macroscopic particles. If they are noncohesive, the forces between them are strictly repulsive so that the shape of the material is determined by external boundaries and gravity. If they are dry, any interstitial fluid, such as air, can often be neglected in determining many but not all of the flow and static properties of the system. Yet despite this seeming simplicity, granular material behaves differently from the other familiar forms of matter—solids, liquids or gases—and might therefore be considered an additional state of matter in its own right.

At the root of the unique status of granular materials are two characteristics: Ordinary temperature plays no role, and interactions between grains are dissipative because of the existence of static friction and the inelasticity of collisions. For example, a sandpile at rest with a slope less than the angle of repose behaves like a solid: The material remains at rest even though gravitational forces create microscopic stresses on its surface. If the pile is tilted several degrees above the angle of repose, grains start to flow. (See the cover of this issue.) However, this flow is clearly not that of an ordinary liquid because it exists only in a boundary layer at the pile's surface, with no movement at all in the bulk. We might view granular flow as that of a dense gas, because gases too are made of discrete particles with negligible cohesive forces between them. In contrast to ordinary gases, however, thermal energy, kT, is completely insignificant in granular materials. The relevant energy scale for a grain of mass m and diameter d is its potential energy mgd, where g is Earth's gravitational acceleration. For a typical sand grain, this energy is at least 10^{12} times kT at room temperature, and ordinary thermodynamic arguments become useless. For example, many studies have shown that vibrations or rotations of a granular material will induce particles of different sizes to separate into different regions of the container.[1–3] Because there are no attractive forces between the particles, this separation would at first appear to violate the principle that entropy must increase, which normally favors mixing. However, in a granular material the fact that $kT \approx 0$ implies that entropy considerations can easily be outweighed by dynamical effects, which now become of paramount importance.

Temperature allows a system to explore phase space. The fact that kT is negligible in a granular material precludes such exploration. Unless perturbed externally, each metastable configuration of the material will last indefinitely, and no thermal averaging over nearby configurations will take place. Because each configuration has its unique properties, it is difficult to achieve reproducibility of granular behavior, even on large scales and certainly near the static limit where friction is important. Another role of temperature in ordinary gases or fluids is to provide a microscopic velocity scale. Again, in granular materials this role is completely suppressed, and the only velocity scale is the one imposed by any macroscopic flow itself. It is possible to formulate an effective "granular temperature" in terms of velocity fluctuations around the mean flow velocity.[4] Yet, as we will see, such approaches do not always recover thermodynamics or hydrodynamics because granular collisions are inelastic.

The science of granular media has a long history. Much engineering literature is devoted to understanding how to deal with these materials. Nevertheless, the technology for handling and controlling granular materials is not as well developed as that for handling conventional fluids. Estimates suggest that we waste 40 percent of the capacity of many industrial plants because of problems

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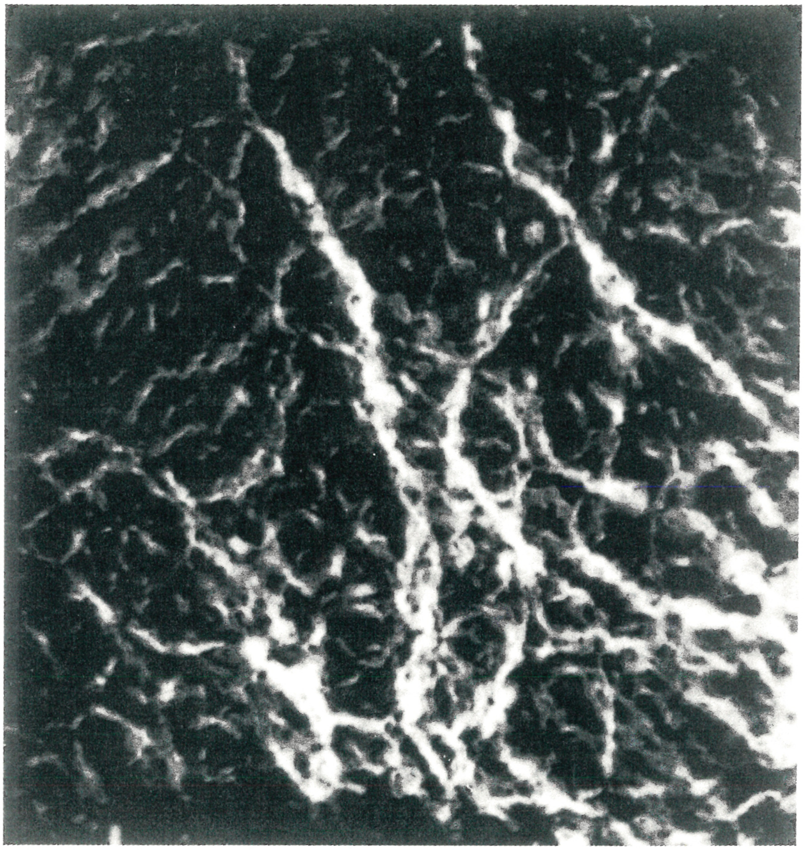
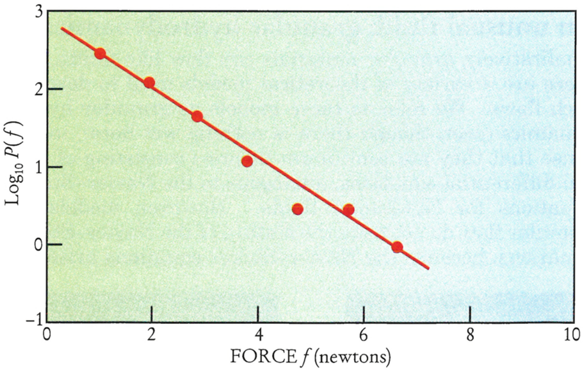


FORCES WITHIN GRANULAR MEDIA. a: Forces between pyrex spheres surrounded by a water–glycerol mixture with a matching refractive index can be visualized when the three-dimensional assembly is viewed between crossed circular polarizers. Stress-induced birefringence makes the beads visible when a force exerted on a piston covering the top surface of the container. b: The forces f exerted by spheres at the bottom of a cylindrical container filled with spheres can be measured by lining the bottom of the container with carbon paper and applying a force to the top surface of spheres. The distribution of forces $P(f)$ decreases exponentially as the force increases, as indicated by the excellent fit to the curve $P(f) = c\exp(-f/f_0)$. (Adapted from ref. 9.) FIGURE 1

b



encountered in transporting these materials from one part of the factory floor to another.[5] Over the last decade there has been a resurgence of interest in this field within physics (for overviews, see reference 6). The science of these materials has clear relevance to what is being done in other areas of condensed matter physics, with the result that sandpiles have become a fruitful metaphor for describing many other, and often more microscopic, dissipative dynamical systems. One finds slow relaxations in vibrated sandpiles that resemble those found in glasses, spin glasses and flux lattices. Fluidlike behavior, similar to that exhibited by conventional liquids, can be induced[7] in these materials. Some of the observed nonlinear dynamical phenomena are relevant to breakdown phenomena in semiconductors, stick–slip friction and earthquakes. A recent, powerful use of sand as a metaphor has been the idea of self-organized criticality, originally described in terms of avalanches in a sandpile.[8] This self-organization paradigm was postulated to have applicability to a wide realm of natural phenomena.

The following three sections explore the unique behavior of granular materials and contrast it with that of ordinary solids, liquids and gases.

An unusual solid: sand at rest

Even in the resting state, granular materials exhibit a host of unusual behaviors. For example, when granular material is held in a tall cylindrical container, such as a silo, the pressure head is not height-dependent as it would be in a normal fluid; that is, the pressure at the base of the container does not increase indefinitely as the height of the material inside it increases. Instead, for a sufficiently tall column, the pressure reaches a maximum value that is independent of the height. Because of contact forces between grains and static friction with the sides of the container, the container walls support the extra weight.

We can investigate the network of forces within the pile in greater detail. Figure 1a shows the stresses in a three-dimensional arrangement of particles. The forces appear to be very heterogeneous, forming chains along which the stresses are particularly intense. These chains play an important part in many of the properties of the granular material, such as the transmission of sound.[6]

The distribution of forces within the pile is not clear from figure 1a alone. It can be found by simply placing a piece of carbon paper in the bottom of the container and measuring the areas of the marks left by the forces, f, of the individual beads on the paper. The distribution of forces is

$$P(f) = c\exp(-f/f_0) \qquad (1)$$

where c and f_0 are constants.[9] (See figure 1b.) The fluctuations in the force are large and scale with depth in the same way as does the mean force—rather than as its square root as one might initially expect. Such behavior has been explained[9] in terms of a simple model, in which masses placed on a lattice distribute their weights unevenly and randomly to the particles on the layer below them. This model can be solved exactly in a number of cases and, in agreement with experiment and simulation, it yields an exponential distribution of large forces. Shear experiments also attest to the extreme heterogeneity of granular media.[6,11]

A fundamental issue in the physics of granular media concerns their packing. Depending on how the container is filled, a random assembly of spheres can be packed with a volume fraction anywhere from $\eta = 0.55$ to $\eta = 0.64$.[6] Through static friction, force chains can hold the assembly in a metastable configuration between these limits and prevent it from collapsing. How does the system pass between these states? Because kT is negligible, the only way the density can change is as a result of external disturbances of the container, for instance by vibrations.

Recent studies of granular material settling under vibrations indicate that the relaxation in these systems is, in fact, logarithmically slow.[12] Even after 100 000 vibration cycles, a tube filled with granular material may still undergo significant compaction before a steady state is reached. A variety of models have been proposed to account for this extremely slow settling.[13] Perhaps the most plausible explanation is based on the consideration of excluded volume effects.[12] A simple corresponding picture is that of a parking lot without assigned slots, and with a high density of equal-sized, parked cars (or particles). For the person wishing to park an extra vehicle (or insert an extra particle into the bead pack), the all-too-familiar situation is that there exist large, but not quite large enough, voids between objects already in place. How many other cars (or particles) have to be moved just a bit for the additional one to fit in? If all densification occurs by random "parking" and "unparking" events, it takes the cooperative motion of many objects to open up new slots. The resulting approach to the steady-state density is logarithmic in time.

An unusual fluid: granular hydrodynamics

Qualitatively, granular materials can flow like fluids, and there are a variety of theoretical models used to describe such flows. We refer to these models as granular hydrodynamics (even though there is nothing wet here), in the sense that they are continuum theories consisting of partial differential equations, analogous to the Navier–Stokes equations for Newtonian fluids. However, models for granular flow do not have the stature of the Navier–Stokes equations, because the Navier–Stokes equations arise out of an averaging process over length and time scales that are much larger than typical microscopic scales, and much smaller than macroscopic scales—a separation of scales that may not occur in granular flows.

Dense slow flows and rapid gaslike flows are useful idealizations for the development of models. Because real granular systems rapidly dissipate energy, they often simultaneously exhibit both flow types in different spatial domains, and one unresolved question is how to model the transition between the two. When the density is low, kinetic theory models[4] can be used to describe granular materials. To maintain this state, however, energy must be continuously supplied, for instance by shaking.

The other extreme is treated by models for quasistatic plastic deformation, based on Osborne Reynolds's principle of dilatancy, and on the idea that deformations in compacted materials are typically irreversible.[6,10] Dilatancy occurs because the grains interlock under applied normal stress, and the material will begin to deform for shear stresses only above a yield point that is determined by the applied normal stresses. Specific models consist of conservation laws in the form of partial differential equations augmented by constitutive models. Thus there is the standard continuity equation for mass conservation, an energy equation and a momentum equation. The momentum equation is perhaps the most revealing. It relates the stress tensor T_{ij} and the strain rate tensor

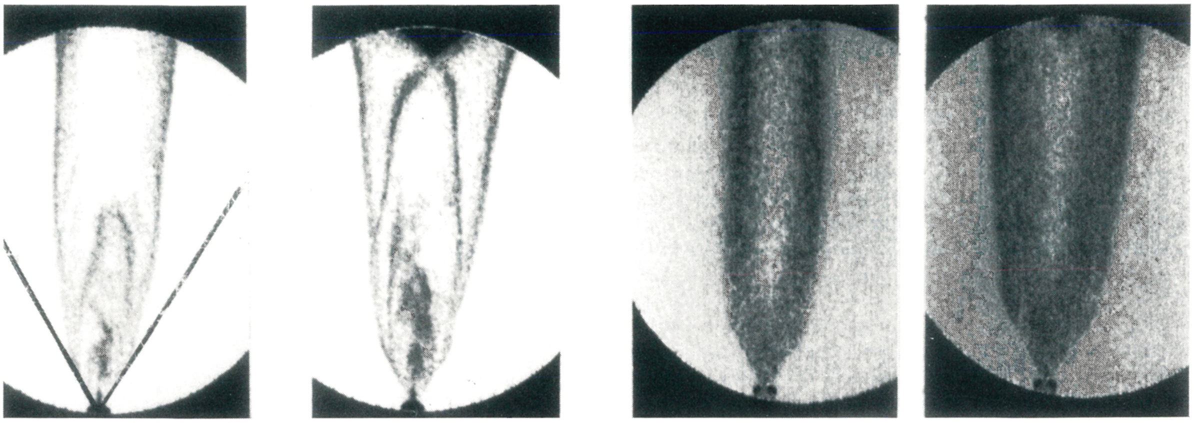
$$V_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \tag{2}$$

where v_i is the ith component of the velocity field. In one of the simplest versions of these models,[10]

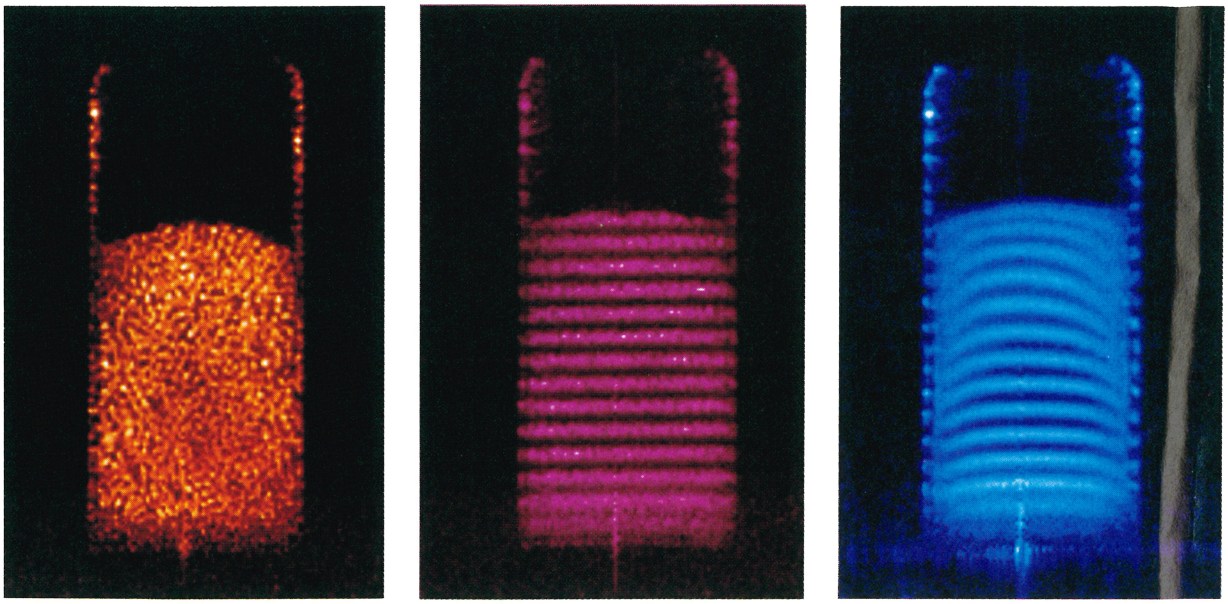
$$T_{ij} = \sigma \left(\delta_{ij} + k \frac{V_{ij}}{|V|} \right) \tag{3}$$

Here, $|V|^2 \equiv \sum V_{ij}^2$ and k is a constant that is characteristic for each material. A comparison of this equation with the Navier–Stokes equations reveals that the ordinary viscous terms, proportional to the viscosity and the velocity gradients, have been replaced by terms independent of the shear rate. This feature is quite remarkable because it implies that an overall increase in velocity leaves the stress unchanged.

Models like that specified by equation 3 are used in soil mechanics and in the design of materials-handling devices, such as hoppers.[6] However, experiments on flow in thin hoppers[14] have revealed dynamic behavior not



X-RAY IMAGES OF WAVES in a hopper of flowing sand show (two images at left) that waves form when the sand is composed of rough grains—in this case, sieved construction sand; darker regions in the image correspond to lower density. When the grains are sufficiently smooth—in this case, Ottawa sand—the waves are absent (two images at right). Typical grain sizes of the two materials were identical, ranging from 0.6 to 0.7 millimeters. (Adapted from Baxter and Behringer in ref. 14.) FIGURE 2 a b c



explained by such approaches. Figure 2 contrasts x-ray images of flow out of a hopper for rough and smooth-grained materials. These experiments show density waves for the rough materials, but not for the smooth, nearly spherical materials. In the case shown here for rough grains, the waves propagate upward, but the propagation direction reverses if the hopper angle is made steep enough. The role of grain shape in these processes is critical and needs to be better understood.

One exciting aspect of the present state of the physics of granular media is the lively debate about the causes of some of the most prominent behaviors that these materials exhibit when vibrated. Two such debates, discussed briefly here, concern the cause of vibration-induced convection and heaping and the cause of vibration-induced size separation.

Although convective flow in vibrated granular material was first observed by Michael Faraday 160 years ago, the underlying mechanisms are at best only partially understood. Both segregation and convection occur when the material is shaken in the vertical direction. Typically the container is shaken sinusoidally, as $z = A\cos(\omega t)$. When the maximum acceleration of the container is greater than the gravitational acceleration—that is when $\Gamma = A\omega^2/g$ is larger than 1—the material rises above the floor of the container for some part of each cycle, dilating in the process, and allowing a macroscopic convection roll that continuously transports grains. In cylindrical or rectangular vessels, the flow is usually upward in the center and downward in

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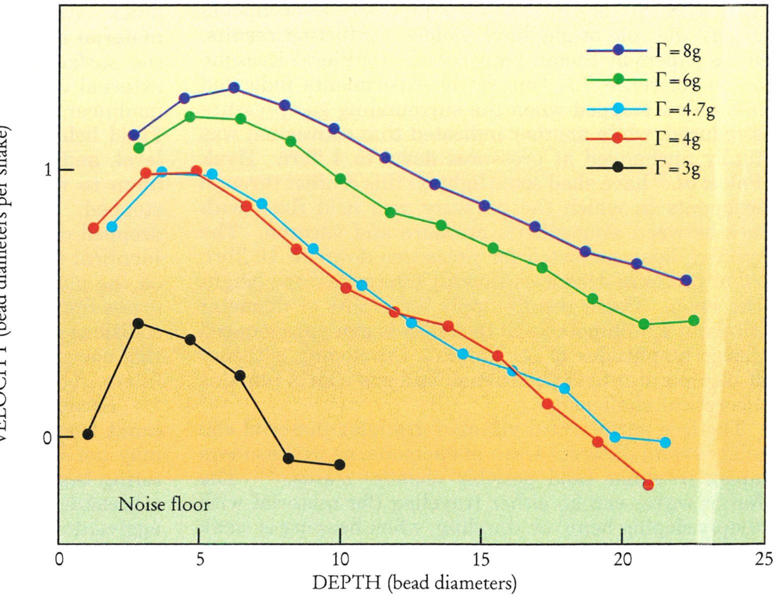


MAGNETIC RESONANCE IMAGES can reveal the behavior of particles hidden deep within a granular medium, in this case a cylinder filled with white poppy seeds. **a**: In this image, each visible grain is an individual seed measuring roughly 1 millimeter across. The coating of seeds glued to the inner surface of the container walls serves as a stationary reference and is visible above the filling level. **b**: When the spatial modulation of the spin polarization is changed to the vertical or z direction, the peaks of the modulation (bright bands) label narrow regions within the granular material. **c**: This image was prepared in the same way as image b, except the container was given a single vertical shake. The curvature of the horizontal stripes directly gives the flow profile of the seeds (displacement per shake) and can be seen by referring to the seeds glued to the container walls. **d**: For a wide range of accelerations Γ, the vertical velocity, v_z, of the central region of each band decreases rapidly as a function of depth, z, below the top surface. In this plot, a straight line indicates an exponential depth-dependence of the velocity. (Adapted from Enrichs *et al.* in ref. 3.) FIGURE 3 a thin stream along the sidewalls, leading to the formation of a central heap,[3,7,15] down which there is a steady avalanche of grains. With different boundary conditions, such as sidewalls that slant outward, it is possible to reverse the sense of the convection roll and induce downward flow in the center.[3,16]

At least two mechanisms have been proposed to explain these rolls for experiments in which the entire layer is shaken uniformly. One mechanism involves friction with the walls of the container. Several experiments and simulations have shown that there is a ratchet effect that produces a thin, rapidly moving boundary layer near the walls and leads to circulating flow.[3] Recent experiments have determined both the depth-dependence and the detailed shape of the convection velocity profile using magnetic resonance imaging to probe noninvasively the granular motion deep inside the material.[3,17] (See figure 3.) The fastest flow occurs in the thin layer near the walls—a situation different from what occurs for a conventional fluid, for which the no-slip condition applies at the walls. This raises a number of issues about what boundary conditions are appropriate for convection and other granular flows.

A second mechanism, first suggested by Faraday, for convection and heaping depends on the presence of interstitial gas. This effect is particularly apparent when friction with the container walls is reduced. Experiments to clarify the role of gas have yielded conflicting results. (See the papers by Stefan Fauve *et al.* and Pierre Evesque *et al.* in reference 7.) One set of experiments indicated that the flow stopped when the surrounding gas pressure was reduced, while another indicated that convection was virtually unchanged at pressures down to 4 Torr. Hyuk K. Pak *et al.*[15] have shed some light on this conflict through experiments in which the pressure was held fixed with values between atmospheric pressure and vacuum. The convective heap persisted for pressures down to 10 Torr. Further pressure decreases steadily diminished the height of the heap. These results apply for grains of diameter up to about 1 millimeter, and the effect is more pronounced for large amplitudes of oscillation. Developing a theory that incorporates both the friction and gas effects remains a theoretical challenge.

The free surface of a vibrated granular material can exhibit several different wave phenomena, as well as more complex irregular (and possibly chaotic) states.[15,7] The different waves can be either traveling (for material with a steeply sloping heap) or standing, when heaping is weak or nonexistent. Figure 4 shows examples of the subharmonic standing waves; well-defined wave patterns and their superpositions occur that are strikingly familiar from Faraday instabilities in ordinary liquids. In the first two parts of this figure, the waves are confined to a narrow rectangular container. Part c shows the striking patterns that evolve when the container is a large open cylinder. (See PHYSICS TODAY, October 1995, page 17.)

Another key feature of vibrated or flowing granular material is its unique mixing and size-separation ("unmixing") behavior. Separation phenomena occur in long, slowly rotating cylinders, where the cylinder axis is horizontal; particles with different dynamical angles of repose aggregate into sharply delineated regions.[1] Conversely, an important question concerns the rate of mixing in horizontally rotating cylinders or drums that are partially filled and start from an inhomogeneous state. In these situations, a steady succession of discrete avalanches cause mixing, and the degree of mixing as a function of filling level can be calculated from geometrical arguments.[18] Both mixing and unmixing bear directly on such technically important processes as the separation of "fines," or dustlike particles (which may or may not be desirable), or the mixing of powdered drugs with a binder, when a well-controlled and homogeneous mixture is essential.

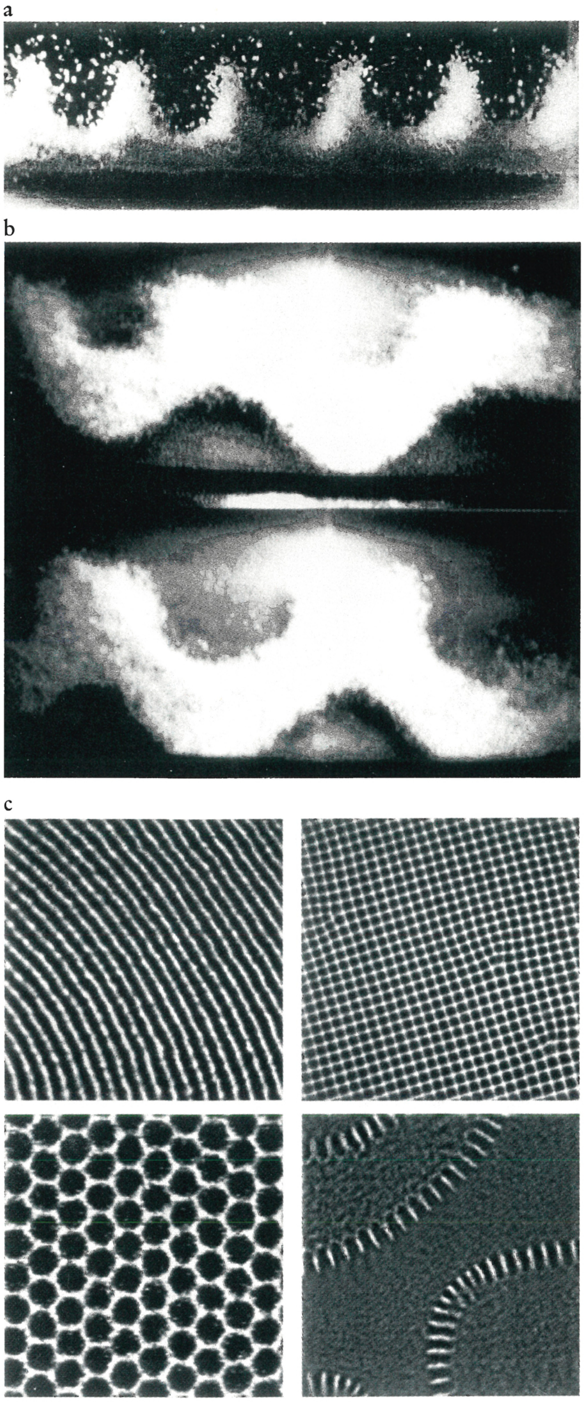
When granular materials are shaken, particles of different sizes tend to separate, with the largest particles moving to the top, independent of their density.[2,3] Several mechanisms have been associated with such size separation, including sifting—in which small particles fall through the gaps between large particles if the gaps are large enough—and local rearrangements—in which large particles are wedged upward as smaller grains avalanche into voids beneath them during the dilation phase of each shake.[2] In vertically shaken systems, however, experiments[3] have shown a direct link between convection and size separation: Large particles become entrained in the upward convective flow, but, once on the top surface, they remain stranded because they cannot follow the smaller grains in the thin layer of downward flow along the container walls.

An unusual gas: inelasticity, clustering, collapse

One critical difference between granular media and ordinary gases and fluids deserves particular attention: Because interactions between grains are inherently inelastic, some energy is lost in each collision. As a result, novel features arise for the statistical mechanics of these systems. Any seemingly fluidlike behavior of a granular material is a purely dynamic phenomenon. For example, the surface waves do not arise as a linear response to external energy input, but are a consequence of a highly nonlinear, hysteretic transition out of the solidlike state. Fluid behavior sets in only above a threshold excitation level, and inelastic collisions will bring the granular medium to rest almost instantly after the energy input is stopped. An individual grain, such as a single marble, dropped onto a glass plate, will bounce for quite a while. Identical marbles, loosely filled into a sack, will stop dead on the plate. This strikingly different collective behavior arises from the exceedingly large number of rapid inelastic collisions among neighboring grains. In fact, we often use this energy absorption in such applications as packaging fillers, recoilless hammers and Hacky Sacks.

Because real granular materials are inelastic, energy input from the boundaries, as in an ordinary heat bath, may not be sufficient to thermalize the system. If clustering begins to occur inside the system, it may indicate a breakdown of Newtonian hydrodynamics because those aggregates will not be able to melt away. Recently, this clustering in the regime of finite inelasticity has become a focus of much interest.[19] The conditions under which clustering should appear have been estimated: If a system of linear extent L is started in a uniform state with grains occupying a volume fraction, η, the solutions provided by Newtonian hydrodynamics become linearly unstable to cluster formation once the product ηL exceeds a constant that depends on the degree of inelasticity. If L is large enough, the system always becomes linearly unstable toward cluster formation, no matter how small the inelasticity of each collision. In figure 5, we clearly see the tendency of inelastic collisions to produce particle clustering. This snapshot is taken from a two-dimensional simulation by Isaac Goldhirsch and Gianluigi Zanetti.[19] Here, a system of hard disks was started with random initial velocities, in the absence of gravity, and without external forcing.

A particularly exciting development has been the recent recognition of a special type of clustering called inelastic collapse. Sean McNamara and William Young[19] showed that inelasticity can lead to an infinite number of collisions occurring in a finite time. In one dimension, such a collision sequence leaves the particles "stuck"



COMPLEX WAVE BEHAVIOR is easily observed in thin layers of granular materials shaken vertically with frequency f and acceleration $\Gamma = A\omega^2/g$. **a**: Side view of a subharmonic standing wave in a quasi one-dimensional container filled with glass beads. ($f = 20$ hertz, $\Gamma = 3.5$.) (From Douady *et al.* in ref. 7.) **b**: A different standing wave in an annular container. The two snapshots are taken at successive oscillation periods of the container and illustrate the subharmonic nature of the response, which repeats after two drive cycles rather than one. (From Pak and Behringer in ref. 7.) **c**: Two-dimensional systems of granular material organize into a variety of stationary subharmonic patterns that depend on f and Γ. These photos show top views of large cylindrical vessels filled with 165-micron-diameter brass spheres to a nominal depth of about 8 spheres. Stripe (upper left) and square patterns (upper right) form for $f \approx 40$ Hz, $\Gamma \approx 3$. Highly curved interfaces (lower right) between essentially flat, featureless plateaus coexist 180° out of phase in different parts of the cell ($f \approx 40$ Hz, $\Gamma \approx 5$). Hexagonal patterns (lower left) occur when the system is driven at two frequencies (16 and 32 Hz) and $\Gamma \approx 2.8$.. (Photos courtesy of P. Umbanhowar.) FIGURE 4

together in close contact with no relative motion.[19] Remarkably, inelastic collapse also persists in higher dimensions, where it produces dense chainlike clusters. The precise relationship between inelastic collapse and the phenomenon of clustering, which is the initial signature of a breakdown of ordinary hydrodynamics, needs clarification. One plausible scenario is that once the system forms clusters, the occurrence of inelastic collapse requires additionally that the energy loss per collision exceed a critical value. Another conceivable scenario envisions all clusters as transients that eventually must terminate in either inelastic collapse or the formation of shear bands.[19]

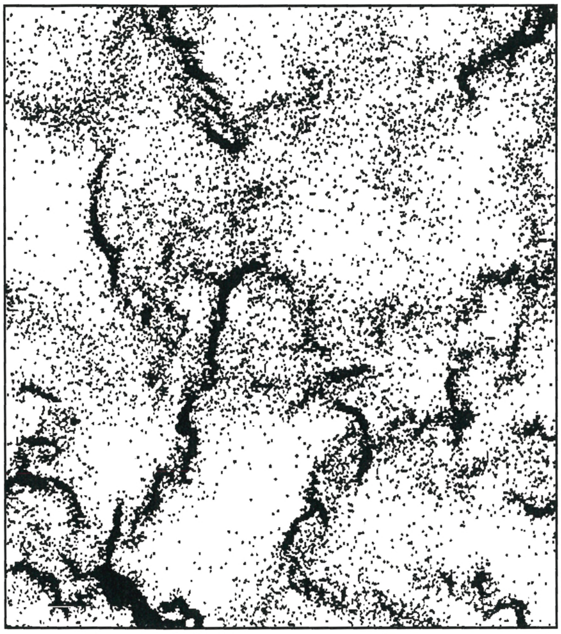
Perhaps the most remarkable aspect of clustering, which is also true in the regime of inelastic collapse, is that it leads to long stringlike grain configurations, rather than to a shapeless blob of particles. As noted by Goldhirsch and Zanetti, figure 5, in this sense, is qualitatively similar to density maps of the visible universe. It is worth speculating that the attractive gravitational potential plays the role of a confining container to keep the density high enough for clusters to form. Hence, on very large scales, the structures created by repeated inelastic collisions may also be responsible for the coagulation observed in "gases" made up of stars or galaxies.

Perhaps the quotation from Victor Hugo at the beginning of this article is not so far-fetched after all. The motion of grains of sand may indeed be relevant to the creation not just of worlds but also of galaxies and the structure and formation of the astronomical landscape.

Scientific challenges ahead

This article has only touched on some of the distinctive properties of granular materials. The preceding sections have shown that, depending on how we prepare and excite these materials, they act as highly unusual solids, liquids or gases. Clearly, the physics of granular materials spans a wide variety of phenomena, with applications ranging from the mundane to the celestial. The array of experimental techniques used to study these systems is also quite broad and spans a considerable range of sophistication—from counting spots left by carbon paper to magnetic resonance imaging and x-ray tomography. Despite their apparent simplicity, these materials display an intriguing range of complex, nonlinear behavior, and explanations of this behavior often appear to challenge existing physics wisdom. Many of the new ideas and techniques being developed within the specific context of granular materials are applicable to a much wider range of systems of similar metastable characteristics and in which the thermal energy, kT, is irrelevant—for example, in foams or in the tunneling regime for superconducting vortex arrangements.

The recognized deficiency of our understanding of granular matter offers tremendous opportunities for new insights from physics to make a strong, technologically relevant impact. However, many scientific challenges must first be resolved. For example, for the case of packing, although we know that the packing history is relevant, we do not know how to include it in theories of



Typical configuration of 40 000 inelastically colliding particles exhibits clustering in two dimensions. Here the coefficient of restitution is 0.6, the time corresponds to 500 collisions per particle and the average area fraction occupied by particles is 0.05. Note the chainlike structures in the photo. (From Goldhirsch and Zanetti in ref. 19.) Figure 5

compaction or stress patterns within the medium. Likewise, when attempting a hydrodynamic approach to granular flow, it is obvious that the ordinary hydrodynamic nonslip boundary assumptions are invalid, but we are still at a loss as to how to treat the boundaries correctly.

It also remains to be seen to what extent Newtonian hydrodynamics must be (or can be) modified to describe granular media correctly in the first place. Certainly the authors of this article debate this issue among themselves. The debate extends to a related question—namely whether the idea of inelastic collapse is more than a beautiful theoretical concept or whether it has real experimental ramifications, and whether the difference between inelastic collapse and the more general case of inelastic clustering is experimentally observable. The authors hope that the recent surge of interest in the basic physics of granular media will produce advances that can not only lead to improved applications for technological processes, but also deepen our understanding of the many related aspects of microscopic and macroscopic physics for which sand has been used as a metaphor.

Because of space limitations, the list of references in this article only provides a glimpse of some of the relevant literature, and omits many important contributions. A more complete account can be found in the full version of this article, which is to appear in the "Colloquia" section of Reviews of Modern Physics. We would like to thank our collaborators for many exciting discussions on the topics mentioned in this article. They include G. W. Baxter, E. Ben-Naim, S. N. Coppersmith, E. van Doorn, E. E. Ehrichs, L. P. Kadanoff, G. S. Karczmar, J. B. Knight, V. Yu. Kuperman, C.-H. Liu, B. Miller, E. Nowak, C. O'Hern, H. K. Pak, D. A. Schecter and T. A. Witten. We are grateful to J. Cina and to W. Young and S. Esipov for a critical reading of the manuscript.

References

1. O. Zik, D. Levine, S. G.Lipson, S. Shtrikman, J. Stavans, Phys. Rev. Lett. **73**, 644 (1994). K. M. Hill, J. Kakalios, Phys. Rev. E **49**, 3610 (1994), and references therein.
2. A. Rosato, K. J. Strandburg, F. Prinz, R. H. Swendsen, Phys. Rev. Lett. **58**, 1038 (1987). R. Jullien. P. Meakin, A. Pavlovitch, Phys. Rev. Lett. **69**, 640 (1992). J. Duran, J. Rajchenbach, E. Clement, Phys. Rev. Lett. **70**, 2431 (1993).
3. J. B. Knight, H. M. Jaeger, S. R. Nagel, Phys. Rev. Lett. **70**, 3728 (1993). E. E. Ehrichs, H. M. Jaeger, G. S. Karczmar, J. B. Knight, V. Yu. Kuperman, S. R. Nagel, Science **267**, 1632 (1995).
4. S. B. Savage, Adv. Appl. Mech. **24**, 289 (1984). P. K. Haff, J. Fluid Mech. **134**, 401 (1983). J. Rheol. **30**, 931 (1986). O. R. Walton, R. L. Braun, J. Rheology **30**, 949 (1986). J. T. Jenkins, in *Nonclassical Continuum Mechanics: Abstract Techniques and Applications,* R. J. Kops and A. A. Lacey, eds., Cambridge U. P., New York (1987), p. 213. C. K. K. Lun, S. B. Savage, J. Appl. Mech. **54**, 47 (1987). C. S. Campbell, Ann. Rev. Fluid Mech. **22**, 57 (1990).
5. B. J. Ennis, J. Green, R. Davis, Chem. Eng. Progress **90**, 32 (1994).
6. *Physics of Granular Media*, D. Bideau, J. Dodds, eds., Les Houches Series, Nova Science, Commack, New York (1991). *Granular Media: An Interdisciplinary Approach*, A. Mehta, ed., Springer, New York (1991). H. Hayakawa, H. Nishimori, S. Sasa, Y.-H. Taguchi, Jpn. J. Appl. Phys. **34**, 397 (1995). H. M. Jaeger, S. R. Nagel, Science **255**, 1523 (1992). S. R. Nagel, Rev. Mod. Phys. **64**, 321 (1992). H. M. Jaeger, J. B. Knight. C.-H. Liu, S. R. Nagel, Mater. Res. Soc. Bull. **19**, 25 (1994). R. P. Behringer, Nonlinear Science Today **3**, 1 (1993). R. P. Behringer, Proc. Mater. Res. Soc. **367**, 461 (1995).
7. S. Fauve, S. Douady, C. Laroche, J. Phys. (France) **50**, 187 (1989). P. Evesque, J. Rajchenbach, Phys. Rev. Lett. **62**, 44 (1989). S. Douady, S. Fauve, C. Laroche, Europhysics Lett. **8**, 621 (1989). H. K. Pak, R. P. Behringer, Phys. Rev. Lett. **71**, 1832 (1993). F. Melo, P. Umbanhowar, H. L. Swinney, Phys. Rev. Lett. **72**, 172 (1993). C. R. Wassgren, C. E. Brennen, M. L. Hunt, preprint. F. Melo, P. B. Umbanhower, H. L. Swinney, Phys. Rev. Lett **75**, 3838 (1995).
8. P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
9. C.-H. Liu, S. R. Nagel, D. A. Schecter, S. N. Coppersmith, S. Majumdar, O. Narayan, T. A. Witten, Science **269**, 513 (1995).
10. R. Jackson, in *The Theory of Dispersed Multiphase Flow*, R. Meyer ed., Academic, San Diego, (1983). E. B. Pitman, D. G. Schaeffer, Commun. Pure Appl. Math. **40**, 421 (1987). D. G. Schaeffer, J. Diff. Eq. **66**, 19 (1987). D. G. Schaeffer, M. Shearer, E. B. Pitman, SIAM J. Appl. Math. **50**, 33 (1990).
11. G. W. Baxter, R. Leone, R. P. Behringer, Europhysics Lett. **21**, 569 (1993). B. Miller, C. O'Hern, R. P. Behringer, preprint.
12. J. B. Knight, C. G. Fandrich, C. N. Lau, H. M. Jaeger, S. R. Nagel, Phys. Rev. E **51**, 3957 (1995). H. M. Jaeger, E. Nowak, E. Ben-Naim, J. B. Knight, S. R. Nagel, preprint.
13. G. C. Barker, A. Mehta, Phys. Rev. E **47**, 184 (1993). D. C. Hong, S. Yu, J. K. Rudra, M. Y. Choi, Y. W. Kim, Phys. Rev. E **50**, 4123 (1994).
14. G. W. Baxter, R. P. Behringer, T. Faggert, G. A. Johnson, Phys. Rev. Lett. **62**, 2825 (1989). G. W. Baxter, R. P. Behringer, in *Two-Phase Flows and Waves*, D. D. Joseph, D. G. Schaefer, eds. Springer, New York, (1990), p. 1.
15. H. K. Pak, R. P. Behringer, Nature **371**, 231 (1994). H. K. Pak, E. van Doorn, R. P. Behringer, Phys. Rev. Lett. **74**, 4643 (1995).
16. Y.-H. Taguchi, Phys. Rev. Lett. **69** 1367 (1992). J. A. C. Gallas, H. J. Herrmann, S. Sokolowski, Phys. Rev. Lett. **69**, 1371 (1992). S. Luding, *et al.*, Phys. Rev. E **50**, 1762 and 3100 (1994). T. Poeschel, H. J. Herrmann, Europhys. Lett. **29**, 123 (1995). M. Bourzutschky, J. Miller, Phys. Rev. Lett. **74**, 2216 (1995). H. Hayakawa, S. Yue, D. C. Hong, Phys. Rev. Lett. **75**, 238 (1995).
17. M. Nakagawa, S. A. Altobelli, A. Caprihan, E. Fukushima, E.-K. Jeong, Experiments in Fluids **16**, 54 (1993).
18. G. Metcalf, T. Shinbrot, M. M. McCarthy, J. M. Ottino, Nature **374**, 39 (1995).
19. O. R. Walton, in *Particulate Two-Phase Flow, Part 1,* M. C. Roco, ed., Butterworth-Heinemann, Boston (1992), p. 884. I. Goldhirsch, G. Zanetti, Phys. Rev. Lett. **70**, 1619 (1993). S. McNamara, V. R. Young, Phys. Rev. E **50**, 28 (1994). Y. Du, H. Li, L. P. Kadanoff, Phys. Rev. Lett. **74**, 1268 (1995). P. Constantin, E. Grossman, M. Mungan, Physica D **83**, 409 (1995). ■