

MY ENCOUNTERS —AS A PHYSICIST —WITH MATHEMATICS

Mathematical physics is not a discipline with its own identity. Rather there is mathematics and there is physics, and their cyclical relationship enjoys periods of cooperation interspersed with periods of mutual indifference. New initiatives leading to rapid development in physics frequently are accompanied by mathematical innovation. Examples from the past are the construction of particle mechanics and differential calculus by Isaac Newton and Gottfried Leibniz, developments in general relativity and differential geometry at the time of Albert Einstein and Hermann Minkowski, and advances in group theory and analysis following the invention of quantum mechanics and quantum field theory.

But the links remain sporadic and, as in any two societies that are separated, the languages develop differently, thereby hindering communication. Also, attitudes about achievement acquire distinct emphases: Physicists prize empirical problem solving and model building, in contrast to the mathematicians' appreciation of rigorously proven theorems. Part of the fun of being a mathematical physicist is discovering how the different terminologies describe the same things and how the distinct goals benefit each other. (See figure 1.)

Early in my career, my attitude about the relation between mathematics and physics and about the utility of the former for the pursuit of the latter was informed by a talk I heard given by Paul Dirac, in the Harvard University Loeb lecture series on physics history. Although I did not take notes, later I found a printed version in which he reiterated his forceful position in favor of mathematics in physics:

The most powerful method of advance [in physics] . . . is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and . . . to try to interpret the new mathematical features in terms of physical entities.

Today there is intense cross-fertilization between mathematics and physics, specifically between geometry and field theory. The contact, initially established through Einstein's theory of general relativity (in which the gravitational field is described in terms of the geometry

Descriptions of the physical universe and proofs of abstract mathematical theorems are sometimes intimately related.

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of space-time), surged again about two decades ago. Some of my own research took place during that new beginning, so I present here a reminiscence in the form of a case history of a physics-mathematics encounter, with com-

ceived opinion.

Kinks and zero modes

By the early 1970s, quantum field theory was very much in favor with theoretical physicists, but the quantized equations continued to resist solution. It then occurred to many people that it would be worthwhile to ignore the quantal nature of the fields and solve the equations as if they described nonlinear, classical dynamical systems. Interesting, localized and nondissipative solutions were found very quickly. They were the kinks in one dimension, relevant to physics on a line; vortices in two dimensions, in planar physics; skyrmions and magnetic monopoles in three dimensions—our physical world. Collectively, such solutions were called “solitons,” the name being taken from applied mathematics. Another class of solutions comprised the “instantons” in four-dimensional space-time.

With colleagues at MIT, I addressed the question of how to extract information on the quantum theory from these classical results—that is, we wanted to determine the quantum meaning of classical fields. We made progress on this problem, and at a certain stage Claudio Rebbi and I decided it was important to study the linear small fluctuations about the nonlinear field profiles of solitons and instantons, and also the coupling of other linear systems, such as fermions, to solitons and instantons.

Thus we were led to linear eigenvalue equations, and we realized that the zero modes, which correspond to vanishing eigenvalues, contain especially important information about the quantum physics. The zero modes of the small-fluctuation equations measure the allowed deformations of the soliton or instanton, and the number of these modes gives the dimension of the moduli space for the solution of the nonlinear equation. (The moduli space consists of the set of all possible distinct solutions. For example, if a soliton solution is described by two parameters—say, its location $\mathbf{a}$ and its size c —then the moduli space will be some region in the space of $\{\mathbf{a}, c\}$ values.) In the fermionic Dirac equation, the eigenvalues measure energy: Positive-energy modes describe quantum particles, negative-energy modes correspond to antiparticles and zero modes—when they exist—signal a degeneracy

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ON REGULAR SOLUTIONS OF EUCLIDEAN YANG-MILLS EQUATIONS

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The number of instantons and the number of zero fermion modes in the field of instanton are calculated.

DEGREES OF FREEDOM IN PSEUDOPARTICLE SYSTEMS[★]

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We show that at least $8n-3$ parameters are required to specify an n -pseudoparticle solution in Euclidean $SU(2)$ Yang-Mills theory.

FRONTISPICES OF TWO PAPERS that prove the same result, once in mathematical style by use of the Atiyah-Singer index theorem, and once in physical style by explicit solution of differential equations. FIGURE 1

that gives rise to unexpected quantum numbers, such as fractional fermion charge.

Rebbi and I were delighted to find precisely such zero modes in the equations we studied, and also to establish their physical consequences. But we were surprised that the existence of these special solutions did not depend on the details of the localized profiles of the background solitons and instantons; instead, only their large-distance behavior mattered. The long-range features characterize the topological properties of solitons and instantons. Such properties are not tied to the detailed shape of a field configuration but characterize the configuration as a whole—for example, by specifying its critical points or its nontrivial behavior at infinity.

With this idea in mind, we began to suspect that the occurrence of zero modes was not an accident of our analysis, but a consequence of having nontrivial topological backgrounds. Such backgrounds have nontrivial, nonvanishing large-distance properties, in contrast to the usual case where behavior at infinity is assumed to be trivial and irrelevant.

The corridors of MIT

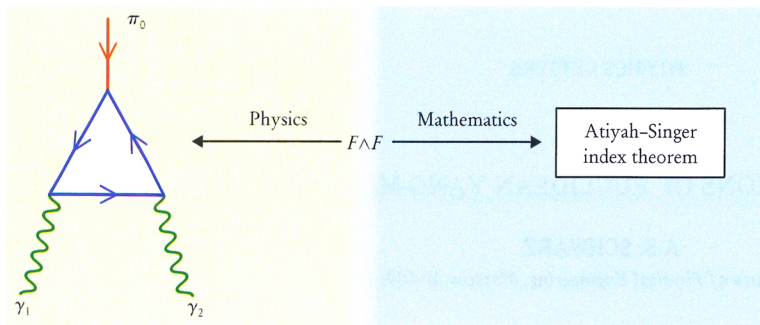
Rebbi and I wanted to find out what mathematicians knew about these ideas. All buildings at MIT are connected, and the math department is in the same structure as my work space. However, locked doors as well as the chemistry department intervene, so communication is ob-

structed. Nevertheless, we walked the corridors of the math department, looking for anyone willing to spend time understanding our questions and answering them in a way comprehensible to nonspecialists, to us physicists. Initially we had no luck, but then we met Barry Simon, who did not have specific information on our problem but suggested that the work of Michael Atiyah and Isadore Singer might be relevant.

Singer had temporarily moved from MIT to University of California, Berkeley, but it so happened that my colleague and collaborator Jeffrey Goldstone knew Atiyah from their student days in Cambridge, England, and had information that Atiyah was coming to visit his mathematics friends in Cambridge, Massachusetts.

So Rebbi and I arranged a meeting in my office. We invited physicists who were working on soliton-instanton questions, and we listened to Atiyah explain how his index theorem with Singer counts instanton zero modes, and how their spectral-flow theorem with V. Patodi is relevant to fractional charge. Here the "index" is the number of zero eigenvalues of a relevant differential operator, and "spectral-flow" describes the passage of eigenvalues across zero, as parameters in the differential operator are varied.

It was especially thrilling for us to learn that the four-dimensional index is given by an integral over the gauge curvature-form F , specifically by $\int F \wedge F$. That was because the integrand, $F \wedge F$, had also arisen in physics papers by John Bell and me, as well as by Stephen Adler,



THE OBSERVED DECAY of the neutral pion into two photons is physically allowed only because of a chiral anomaly that arises in renormalization. Mathematically, the anomaly is given by the wedge product of the electromagnetic field strength, F , with itself. The same quantity is deeply related to index theorems in mathematics. FIGURE 2

as the anomalous divergence of the chiral fermion current, thereby controlling neutral pion decay. (See figure 2.) ("Gauge curvature-form F " is mathematical terminology and notation for what physicists call $F_{\mu\nu}$, the electromagnetic field-strength tensor or its generalizations. Similarly, the wedge " $\wedge$ " denotes a generalized cross product.)

Evidently the chiral anomaly and the index theorem are related; they had been elaborated in the late 1960s at different ends of the same MIT building, by people working in ignorance of each other!

We appreciated very much Atiyah's efforts to make his presentation understandable to us; still, exchanging information was not easy. One young member of the audience impressed Atiyah, who encouraged the fellow to speak because he seemed to understand, better than anyone else, what Atiyah was saying. That person was Edward Witten; as is now well known, he has continued to impress Atiyah and other mathematicians.

Soon thereafter, I was asked to review these exciting new results about quantum field theory at a meeting of the American Physical Society. Since Singer was present, I yielded some of my time to him, with the suggestion that he describe the mathematical connection. But a detailed presentation could not be fit in, so he merely eulogized collaboration between mathematics and physics with an ode:

In this day and age
The physicist sage
Writes page after page
On the current rage
The gauge

Mathematicians so blind
Follow slowly behind
With their clever minds
A theorem they'll find
Duly written and signed

But gauges have flaws
God hems and haws
As the curtain He draws
O'er His physical laws
It may be a lost cause

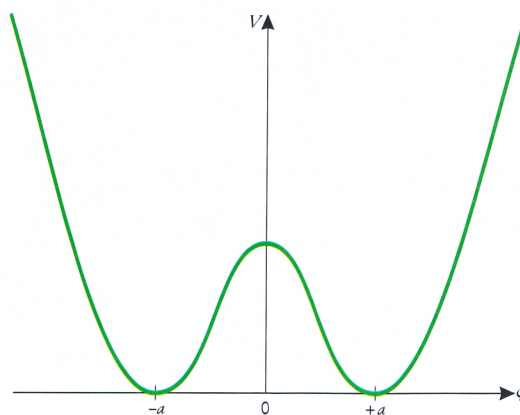
Index theory also received a contribution from physics. The Atiyah-Singer theorem applies to even-dimensional spaces on which a connection is defined. ("Connection" is a mathematical name for the physicist's gauge potential.) However, physicists are also interested in odd-dimensional spaces—where one-dimensional kinks or three-dimensional skyrmions and monopoles reside. These configurations can lead to zero modes, even in the absence of a gauge connection. So we asked the mathematicians about

odd-dimensional index theorems, but apparently they knew nothing about them. At that time I had a mathematically minded student, Costas Callias, and I asked him to prove such a theorem. He achieved success, which then prompted Raul Bott and Robert Seeley to publish a mathematical exegesis of the result, immediately following Callias's paper in *Communications in Mathematical Physics*. Since then I have been happy to see the "Callias index theorem" used and cited.

Shakespeare and Churchill

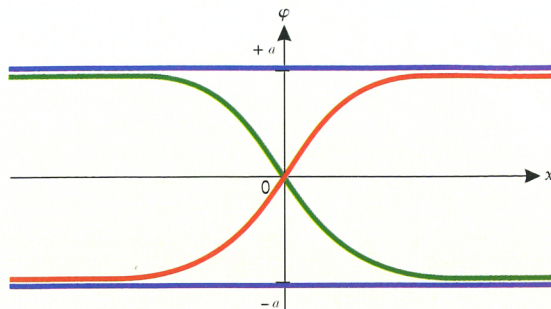
The two approaches to solving problems—the explicit, goal-oriented methods of the physicists and the general theorems of the mathematicians—are well illustrated by the determination of the dimensionality for instanton moduli space: the n -instanton $SU(2)$ solution depends on $8n - 3$ parameters. This result appeared twice in the same issue of *Physics Letters*, once in a paper by Albert Schwarz, who used the Atiyah-Singer theorem, and a few pages later in a paper by Rebbi and me—we solved differential equations to find explicitly $8n - 3$ zero modes. (See figure 1.)

Gauge theories in general and instantons in particular continued to interest mathematicians. They established



THE POTENTIAL IN $\lambda\phi^4$ THEORY. The theory consists of a single scalar field ϕ with a ϕ^4 interaction term and a negative mass-squared term multiplying ϕ^2 . The resulting potential has two minima or vacuum states, at $\phi = \pm a$. This model has long been studied by mathematical physicists and has a physical realization in linear polyacetylene chains, which exhibit fractional charge phenomena. More complicated models involving gauge fields relate to knot theory and the classification of manifolds in mathematics. FIGURE 3

CLASSICAL SOLUTIONS of the $\lambda\phi^4$ theory in 1 space and 1 time dimension: Each of the two vacua (blue, offset slightly for clarity) has trivial topology, being a constant value throughout space. Soliton solutions such as the kink (red) and antikink (green) also exist. These are topologically nontrivial, as they approach different values of ϕ at spatial $+\infty$ and $-\infty$; a kink cannot be smoothly transformed into an antikink or either of the vacua by a finite process. FIGURE 4



the topological properties of the instanton moduli space and produced a construction—but not an explicit formula—for the general solution. The most general explicit expression was given by Craig Nohl, Rebbi and me, but it does not exhaust all the parameters.

Further developments on four-dimensional gauge fields led mathematicians to Donaldson theory. In three dimensions, the Chern–Simons term—a gauge structure that Steven Templeton and I, as well as Jonathan Schonfeld, introduced to physics—has been related by mathematicians to knot theory on manifolds with various topologies, while physicists have applied this term to experimental phenomena on the plane, such as the Hall effect. (See *PHYSICS TODAY*, March 1995, page 17, for developments involving Donaldson theory and physics.)

These parallel investigations by physicists and mathematicians also highlight our differences: We physicists use mathematics as a language for recording observations about physical systems, and this limits our interest in what fascinates mathematicians—the full range of mathematical possibility. For example, the general instanton solution does not appear to be physically relevant; only the original one-instanton and the explicit, but limited, multi-instanton solutions have illuminated physical theory. Even the physicists' language need not always be mathematical. The fractional charge phenomenon, which Rebbi and I inferred from zero modes or from spectral flow, was also independently established by Wu-Pei Su, J. Robert Schrieffer and Alan Heeger, who found a physical realization in linear polyacetylene chains. One of their derivations uses the pictorial language of chemical bonds, in which the only mathematics consists of counting! (See *PHYSICS TODAY*, July 1981, page 19, for a discussion of fractional charge.)

An analogy comes here to mind. The English language contains over 200 000 words, and all of them interest the lexicographer. For William Shakespeare 20 000 different words sufficed to express his ideas in plays and sonnets, whereas Winston Churchill used fewer than 2000 different words in his historically decisive speeches. Physicists, like Churchill, can achieve their goals by making effective use of a limited vocabulary.

Nature's chosen subset

The process of gaining knowledge goes through the same steps in both physics and mathematics. First there is the intuition/guess, then follows the proposal/conjecture and finally comes the verification. For the mathematician, verification consists of constructing a proof, establishing a theorem according to rules whose legitimacy evolves slowly under the direction of the entire mathematics community. But the physicist verifies his ideas by finding a physical correlative: Neutral pion decay validates chiral anomalies (see figure 2), properties of solitons in polyacetylene establish fractional charge (see figure 3). Within physics the rules for giving a proof are constantly and rapidly changing—presuppositions can become modified, experimental facts can evolve.

Because the terms “proof” and “theorem” carry intellectual prestige and pleasure, occasionally there are at-

tempts to employ them in physics. To my mind such efforts are mostly futile and sterile. For example, physicists wanted very much to combine internal and space-time symmetries in a nontrivial fashion and were not daunted by a proven “impossibility theorem.” Rather the “theorem” was circumvented by the simple device of replacing commutators with anticommutators and by grading the algebra. Thus was born supersymmetry, which now is also influencing mathematics. Similarly, when field-theory constructivists “proved” the existence of quantum $\lambda\phi^4$ theory in 1+1 dimensions, they were correct but missed the entire quantum soliton phenomenon, which is the only physically interesting feature of that model. (See figures 3 and 4.)

These days I believe that a statement by C. N. Yang accurately describes physicists' historical use of mathematics:

Physics is not mathematics, just as mathematics is not physics. Somehow nature chooses only a subset of the very beautiful and complex and intricate mathematics that mathematicians develop, and that precise subset is what the theoretical physicist is trying to look for.

This conservative view of mathematics differs from Dirac's radical advice (cited on page 28) that physicists should “try to interpret . . . mathematical features in terms of physical entities.”

Nowadays, however, when we are facing the absence of new experimental data about fundamental phenomena, particle physics theory—as realized in the string program—is driven by mathematics in the manner advocated by Dirac. That was not the way things worked in the past, not even for Dirac. When first confronting his negative-energy solutions, he identified them with the proton—then the only known positively charged particle. As a physicist he did not at first trust his mathematics enough to postulate the existence of the positron!

Thus a question about the future development of physics remains: Shall we succeed in obtaining further and deeper understanding of nature by theoretical, mathematical reasoning alone, or must we wait for new experimental, physical discoveries? New ideas of physics that occupy many people today are built entirely on mathematics, indeed they help create new mathematics. I am immensely curious to learn the ultimate fate of this activity and to see whether it succeeds.

This article is adapted from my Heineman Prize acceptance speech at the April 1995 meeting of the American Physical Society in Washington, DC. Now as then, I acknowledge the colleagues who contributed to the research recognized by the prize: John Bell on quantum anomalies, Steven Weinberg on thermal field theory, Claudio Rebbi on quantum topological effects, and Stanley Deser, Gerard 't Hooft and So-Young Pi on planar field theories. ■