Topological Ideas and Fluid Mechanics

New mathematical techniques and greater computational power have made it possible to apply knot theory and braid theory to fluid flows.

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The use of topological ideas in physics and fluid mechanics dates back to the very origin of topology as an independent science. In a brief note in 1833 Karl Gauss, while lamenting the lack of progress in the "geometry of position" (or *Geometria Situs*, as topology was then known), gives a remarkable example of the relationship between topology and measurable physical quantities such as electric currents.[1] He considers two inseparably linked circuits, each of them a copper wire with ends joined, and flowing electric current. Without comment he puts forward a formula that gives the relationship between the magnetic action induced by the currents and a pure number that depends only on the type of link, and not on the geometry. This number is a topological invariant now known as the linking number. The formula, as well as the very first studies in topology done by Johann Benedict Listing in 1847, became known to Kelvin (then William Thomson), James Clerk Maxwell and Peter Guthrie Tait in Britain.

Hermann Helmholtz's 1858 paper on vortex motion made it possible to apply the new topological ideas to fluid mechanics. His laws of vortex motion state that in an ideal fluid (where there is no viscosity) vortex structures live forever: Two closed vortex rings, once linked, will always be linked. Kelvin, like many others, was in search of an ultimate theory of matter. Tait's translation of Helmholtz's paper on vortex motion provided a wonderful inspiration. Kelvin was so impressed by Helmholtz's laws that he became a fervent believer in the eternal existence of vortex atoms as fundamental constituents of nature. In his theory, atoms were thought to be tiny vortex filaments embedded in an elastic-like fluid medium, called ether. The infinite variety of possible chemical compounds was given by the endless family of topological combinations of linked and knotted vortices. Even the fluid ether surrounding these vortices could have a complex topology in which empty holes and inaccessible closed channels were present. He wrote about this with incredibly imaginative enthusiasm, describing in great detail the physical implication that a topologically complex ether would have.[2]

Kelvin's revolutionary idea to describe fundamental physics through topological properties not only motivated the first studies on the existence and stability of knotted vortices (1875), but also stimulated the interest of many of his most distinguished colleagues and friends.[3] Whereas Tait never succeeded in experimentally reproducing knotted vortex rings, Kelvin's ideas of vortex atoms motivated Tait to produce the first knot table, similar to the modern atom's periodic table. His mathematical classification of knots and links became a fundamental piece of work (see figure 1).[4] Influenced by Maxwell's appreciation of Gauss's and Listing's ideas, Tait tried in 1876 to measure by electromagnetic means topological properties such as knot type, which he called "beknottedness."

But it was Maxwell, more than any other, who truly saw the physical implications of topology.[5] The whole preface of his *Treatise* on electricity and magnetism is permeated by topological ideas. He develops Listing's original ideas of multiply connected regions to study the relationship of electricity and magnetism to forces and potentials. Maxwell notices that if we express a locally conservative force as the gradient of a potential function, then that function will be well defined—that is, single-valued—only inside a simply connected region. (An example of a simply connected region in the plane is a circular patch, while a doubly connected region is a patch with a hole in it.) Moreover, he gives a remarkable example of a particular case of Gauss's linking formula for linked magnetic tubes in the section devoted to magnetism.

While Kelvin's dream of explaining atoms as knotted vortex rings in a fluid ether never came to fruition (despite remarkable analogies with modern string theory), his work was seminal in the development of a topological approach to fluid dynamics. When Leon Lichtenstein published his 1929 book on hydrodynamics, two of the eleven chapters were dedicated to topological ideas.[3] But the difficulty of an immediate application and testing of these ideas limited for many years the use of topological concepts. In recent years, the application of modern results from topology and knot theory and greater access to direct numerical simulation of fluid flows have led to new developments in the qualitative study of fluid mechanics.[6] In this article we give a few examples that show how knot theory and braid theory provide valuable information on fluid mechanical problems.

Links and knots

Knotted and linked structures are ubiquitous in nature and in fluid flow in particular. Their scale lengths range from 10^{-10}–10^{-6} m for superfluid vortices; to 10^{-2}–10^{2} m for fluid eddies, vortex filaments and tornadoes; and may reach 10^{6}–10^{10} m as in the case of magnetic flux tubes,

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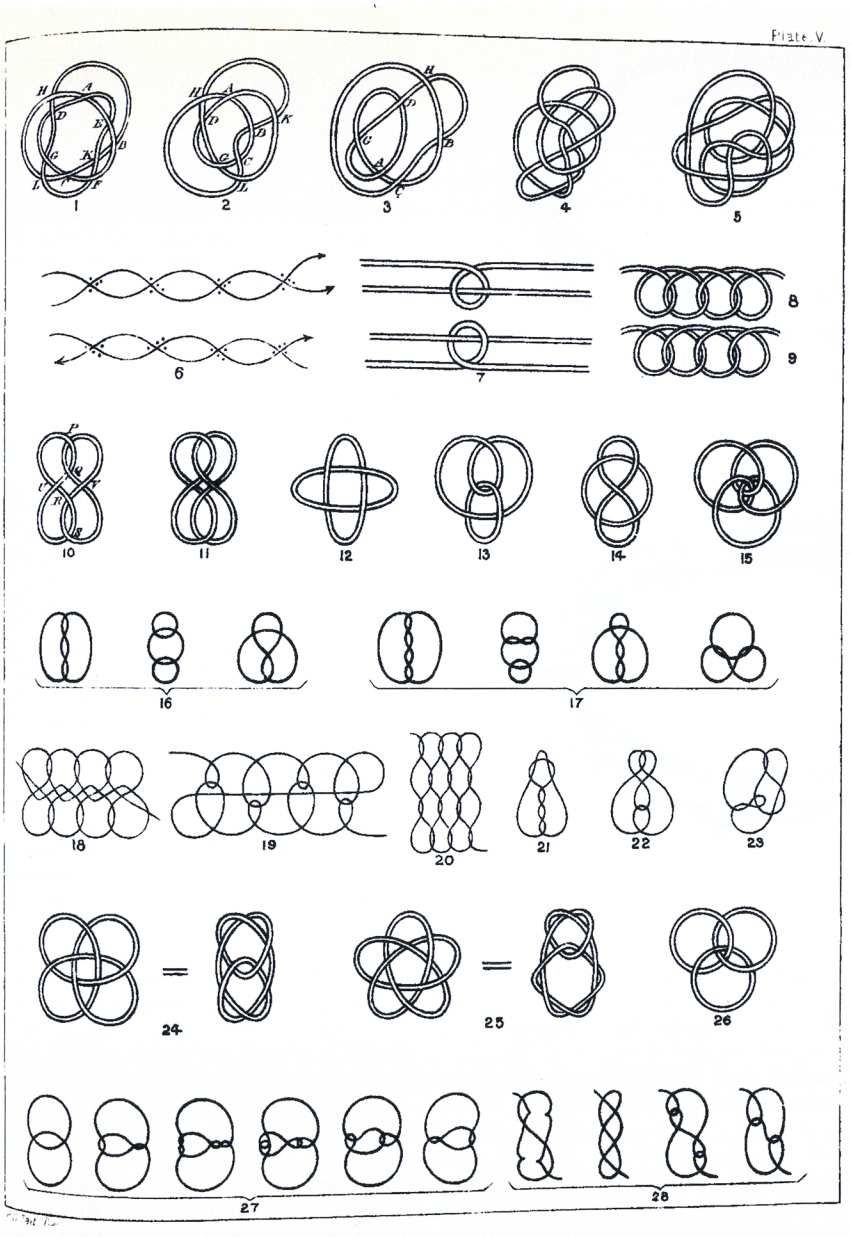
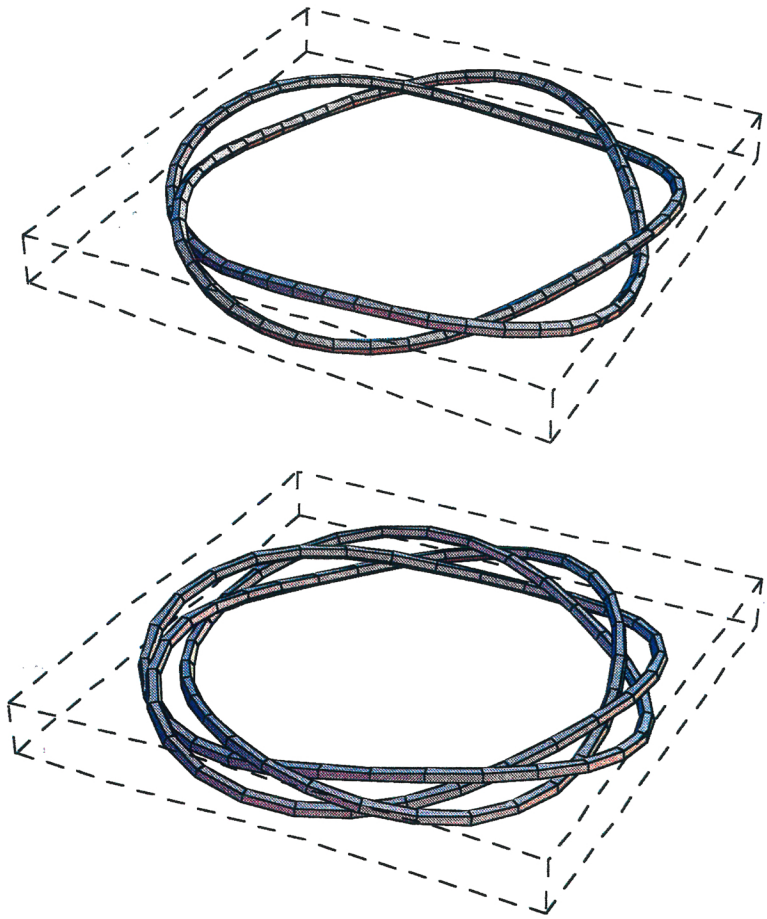


KNOTS AND LINKS, diagrammed by Peter Tait[4] and first published in 1877. Tait calculated the linking number Lk, which he called "belinkedness," of the two-component links 10–14 (third row). The magnitude of the linking number is that of one-half the sum of the signs of the crossings in the link diagram. One assigns a direction to each curve and uses a right-hand rule to assign $+1$ or -1 to crossings. For example, if a line going from south to north crosses over a line going from east to west, one assigns $+1$ to the crossing. Every link can be projected to the plane with a minimum number of crossings, so that Lk is a topological invariant of the link type. Links 10, 12 and 13 each have $Lk = \pm 2$, whereas links 11 (equivalent to two unlinked rings) and 14 have $Lk = 0$ (disregarding the self-crossings of each link component). The fact that non-trivial links may have $Lk = 0$ was actually first discovered by Maxwell in his study of magnetic links such as link 14 (see reference 5, volume 2, article 421). Tait gave a physical explanation for this and studied magnetic effects induced by currents flowing in linked wires in his (unsuccessful) attempt to measure topological properties experimentally. FIGURE 1

plasma loops and magnetic arches in stellar atmospheres. Observational and experimental evidence of complex braided and entangled fluid structures is now well documented in the literature, in the context of both classical fluid mechanics and magnetohydrodynamics.[7]

The presence of tube-like structures at different length scales seems to be a generic feature of organized fluid patterns. Although in real fluids these structures may be rather evanescent (because of dissipative effects), their lifetime can be long enough for them to transport physical properties efficiently throughout the fluid, making them important mediators between different stages of fluid evolution. If there is strong coherency, and if motion is little influenced by dissipative forces, then their dynamics can be crudely modeled by ideal fluid mechanics—that is, by Euler's equations. Tube-like structures such as vortex filaments and magnetic flux-tubes are indeed mathematical idealizations. They may be thought of as a bundle of cooked spaghetti (representing vortex lines or magnetic lines) that pressure gradients keep bound together in a tubular shape.

In an ideal fluid there are no dissipative effects; this means that fluid structures cannot diffuse or die out freely. A change in the fluid pattern due to physical recombination or reconnection of fluid structures cannot take place without viscous or resistive effects. Therefore, vortex or magnetic line topology is frozen in the ideal fluid while the structures of these objects, in continuous motion, can be highly distorted by the background flow. This means that if these tubes are initially knotted or linked, they will evolve and deform in the ideal fluid by preserving the type of knot or link that ties them together, even though their geometry may become utterly complicated (think of the difficulty of disentangling unknotted tele-



TORUS KNOTS representing thin vortex filaments. Torus knots (generally denoted by $T_{p,q}$, with p and q relatively prime integers) are non-self-intersecting closed curves wrapped around a mathematical torus (such as a ring) p times in the longitudinal direction (along the large circumference of the ring) and q times in the meridional direction (along the small circumference). The knots shown here are $T_{2,5}$ (**a**) and $T_{3,7}$ (**b**). Such thin vortices are steady solutions of ideal fluid mechanics: Like vortex rings, they translate and rotate in the fluid without change of shape.[11]
FIGURE 2

phone cords). This is a fundamental, intrinsic property of the governing equations (the Cauchy equations), whose topological implications were studied in great detail by Lichtenstein[3] back in the 1920s. Topological properties of ideal fluids are therefore flow-invariant,[8] and physical information expressed in pure topological terms is therefore bound to be conserved as well.

That Euler's equations conserve certain physical quantities such as kinetic energy, linear and angular momentum and vortex strength is well known. In the late 1960s, a seminal work of Henry Keith Moffatt,[9] followed by a series of other contributions,[10] established new fundamental connections between ideal fluid mechanics and topology. This work is based on the topological interpretation of a new fluid invariant, known as helicity. Under Euler's equations the helicity of a vortex tube of vorticity ω and velocity $\mathbf{u}$ is defined by

$$H = \int \mathbf{u} \cdot \boldsymbol{\omega} \, dV \tag{1}$$

The integral is taken over the tube volume V occupied by ω. Now, for n knotted and linked vortex tubes, each of (constant) strength (total vorticity) Φ_i $(1 \le i \le N)$, the helicity of the whole system can be expressed in terms of linking numbers Lk_{ij} as

$$H = \sum_{ij} Lk_{ij} \Phi_i \Phi_j \tag{2}$$

Lk_{ii} (the case of self-linking, when $i = j$) is none other than the linking number of the ith vortex tube, which may be knotted and twisted in the fluid. Lk_{ij}, which is equal to Lk_{ji}, is a topological invariant whose value does not change under continuous deformation of the fluid structure. Since helicity and flux-tube strength are measurable conserved quantities, equation 2 provides useful information about the topology of the flow field. This equation is indeed applied to analyze flow structures.[9] By direct measurements of helicity and application of conservation of topology, one can estimate average geometric quantities, such as the mean twist of field lines, and their contribution to the total energy. Moreover, the magnetic analog of kinetic helicity is particularly important for the study of twisted magnetic fields in plasma physics and tokamak fusion.

Since J. J. Thomson's work on linked vortices,[3] some progress has been made regarding the existence of vortex knots, as part of more general research on dynamical systems exhibiting topologically complex patterns. These include vortex filaments in the shape of torus knots.[11] One of the easiest ways to construct these knots is to wrap a string around a doughnut, or torus. Let the string go around the doughnut q times the short way around, and p times the long way around, then join the two ends of the string. Now eat the doughnut, leaving the string behind—the string will then form the torus knot $T_{p,q}$. (See figure 2.) Vortex filaments in the shape of torus knots move through the fluid in a remarkably simple manner: They translate in the fluid at a constant velocity while rotating like a rigid body. Although these vortices represent solutions of ideal fluid mechanics, their study provides useful information on the stability and evolution of fluid structures.

Braids

A geometric braid is a set of N intertwined curves stretching between two parallel planes. The curves can be specified by *two* coordinate functions of height z: $[x_i(z), y_i(z)]$; $i = 1, \ldots, N$; $0 \le z \le L$. A knotted curve, on the other hand, requires three coordinate functions of arc-length. From this viewpoint, braids are simpler objects than knots and links and hence provide a promising starting point for studies of topological aspects of fluids.

Two geometric braids are topologically equivalent if one can be deformed into the other by motions that keep the two boundary planes fixed. Topological equivalence classes of braids can be readily classified using group theory, whereas the classification of knots is still a difficult problem.

Suppose we replace the letter z by t. The braid becomes $[x_i(t), y_i(t)]$; $i = 1, \ldots, N$; $0 \le t \le T$, a time history of the motions of N particles moving in two dimensions. Braids as time histories are useful in the study of dynamical systems. For example, a braid can represent the paths of the particles in a two-dimensional N-body simulation. Alternatively, a braid can describe the intertwining of a set of phase curves $[x_i(t), \dot{x}_i(t)]$ in a one-dimensional dynamical system.

There are astrophysical applications as well. Magnetic features on the surface of the Sun (small but intense concentrations of magnetic flux) can random walk about each other due to the turbulent convection below the surface. A braid representing the time history of these motions provides topological information about the magnetic field above the surface—in the solar corona. In active regions, where most flares take place, the magnetic field lines emerging from one surface element loop through the corona, only to plunge down into another surface element. (See figure 3.) As these elements move about each other, the loops above become entangled with the same topological structure as the time history braid.[12] The coronal field, of course, cannot keep on tangling forever. Violent reconnection events break down the topological structure. These events may be observed on Earth as tiny flares (microflares) and probably play an important role in larger flares and in the gigantic coronal mass ejections.[7,12] They may also be the source of heat keeping the corona at 2×10^6 K.

The supply of heat to solar and stellar coronae presents a difficult problem for astrophysicists. The heating rate due to the reorganization of magnetic fields depends on the rate of topological entanglement at the photosphere, and on the saturation level of the coronal magnetic field—that is, the level at which magnetic reconnections on average remove structure at the same rate as the input.

Both the commutativity of twist and the relaxation of braids into minimal patterns have relevance here. (See the box below.) An important part of the topological structure comes from twisting caused by vortical motions at the star's photosphere. Because twist commutes with other structures, opposite senses of twist on a tube can cancel. Random vorticity (sometimes positive, sometimes negative) yields only a root-mean-square twist, corresponding to a magnetic energy growing linearly in time. The power input is then independent of time, and thus essentially independent of the time needed to reach saturation. This would be nice for the theorist, but unfortunately the amount of photospheric vorticity needs to be quite high to match observed heating rates.

More complex braid structures generally do not com-

Pigtails and magnetic fields

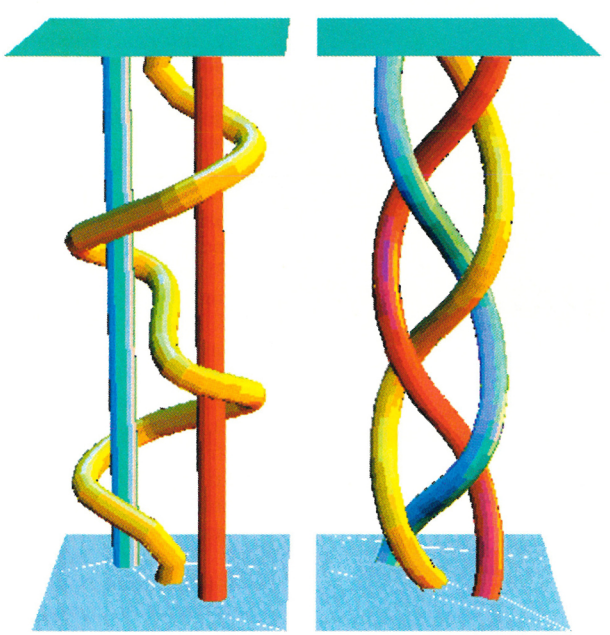
In many situations a set of vortices or magnetic field lines will intertwine about each other, while staying more or less parallel. A topological description of the intertwining involves braid theory. Braids, like knots, have a long history in human technology. Straw hats, for example, are made from long strips of intertwined pieces of straw. Some areas of 19th century rural England had their own distinctive braid patterns, learned assiduously by the young girls employed in the straw industry. Perhaps the most common use of braids today is in plaiting hair into pigtails.

Even simple pigtails have interesting mathematical features. As an illustration, suppose that it becomes fashionable among students to braid their hair using ever more elaborate patterns. As the competition to find novel patterns heats up, two mathematical properties would become apparent.

First, uniform twists commute with all other braid structures. The ends of a pigtail are bound together by a band or ribbon that is free to rotate; adding a uniform twist is useless because it can just travel to the end and disappear. For this reason, pigtails contain at least three bunches of hair (two braided curves can only be twisted).

Second, there are special braid patterns of minimal complexity. A student plaiting her hair according to the first pattern in the figure may be disappointed to discover, after a few hours, that her hair has rearranged itself into the passè standard pigtail (second pattern).

Both the commutativity of twist and the relaxation into minimal patterns are important in the study of magnetic braids.



TOPOLOGICALLY EQUIVALENT braids. Each braid is made of three strings and has an apparently different number of string crossings. By simply rearranging the strings, without moving the end points of the braid, the braid on the left can be transformed to the one on the right, which gives the standard pattern for pigtails and possesses the minimum number of crossings (six).

mute, and hence do not totally cancel. However, structures do have some freedom to relax to minimal patterns, corresponding to minimum energy states, or equilibria, of the magnetic field. Researchers have recently determined lower bounds on equilibrium energy for given measures of topological complexity based on numbers of crossings of braided structures.[6,13] These bounds are expressed by relationships of the kind

$$E_{\min} \geq f(C_{\min}, \Phi, V, N) \qquad (3)$$

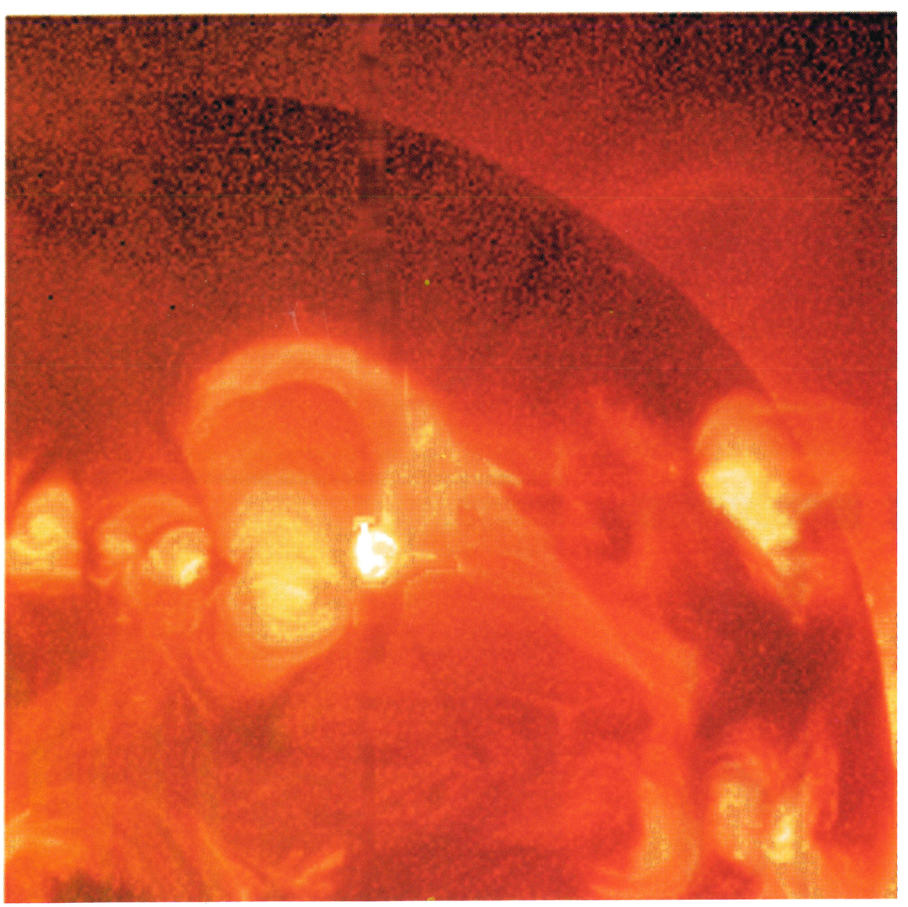
where $E_{\min}$ is the equilibrium energy and f gives the relationship between physical quantities—such as total flux Φ, number of tubes N, magnetic volume V—and topology, given here by the minimum possible number of crossings $C_{\min}$. These relations offer numerous advantages due to the explicit dependence on qualitative properties of the flow field. A simple example is provided by the analysis of three braids, which confirms a prediction by Eugene N. Parker of the University of Chicago that magnetic energy grows *quadratically* in time due to random braiding. This means that heat input associated with the magnetic field depends *linearly* on the saturation level. The heating rates predicted for reasonable guesses as to the saturation level seem to be consistent with astrophysical observations. For more complex topologies (highly tangled fields), though, finding a magnetic equilibrium presents considerable computational difficulties.

A hybrid twisting and tangling model may yet be the most efficient. For solar coronal loops (figure 3), for example, rotation of individual photospheric magnetic elements proceeds more quickly than the braiding of several elements; but as mentioned earlier, twist tends to cancel. However, magnetic elements at the photosphere do not last forever. Sometimes the flux within an element can break loose and wander across the photosphere, later to combine with other free flux to be concentrated into new magnetic elements. If the elements break up and reform before the sense of vorticity changes sign, then the twist will be trapped within a more complicated braid structure, preventing cancellation.

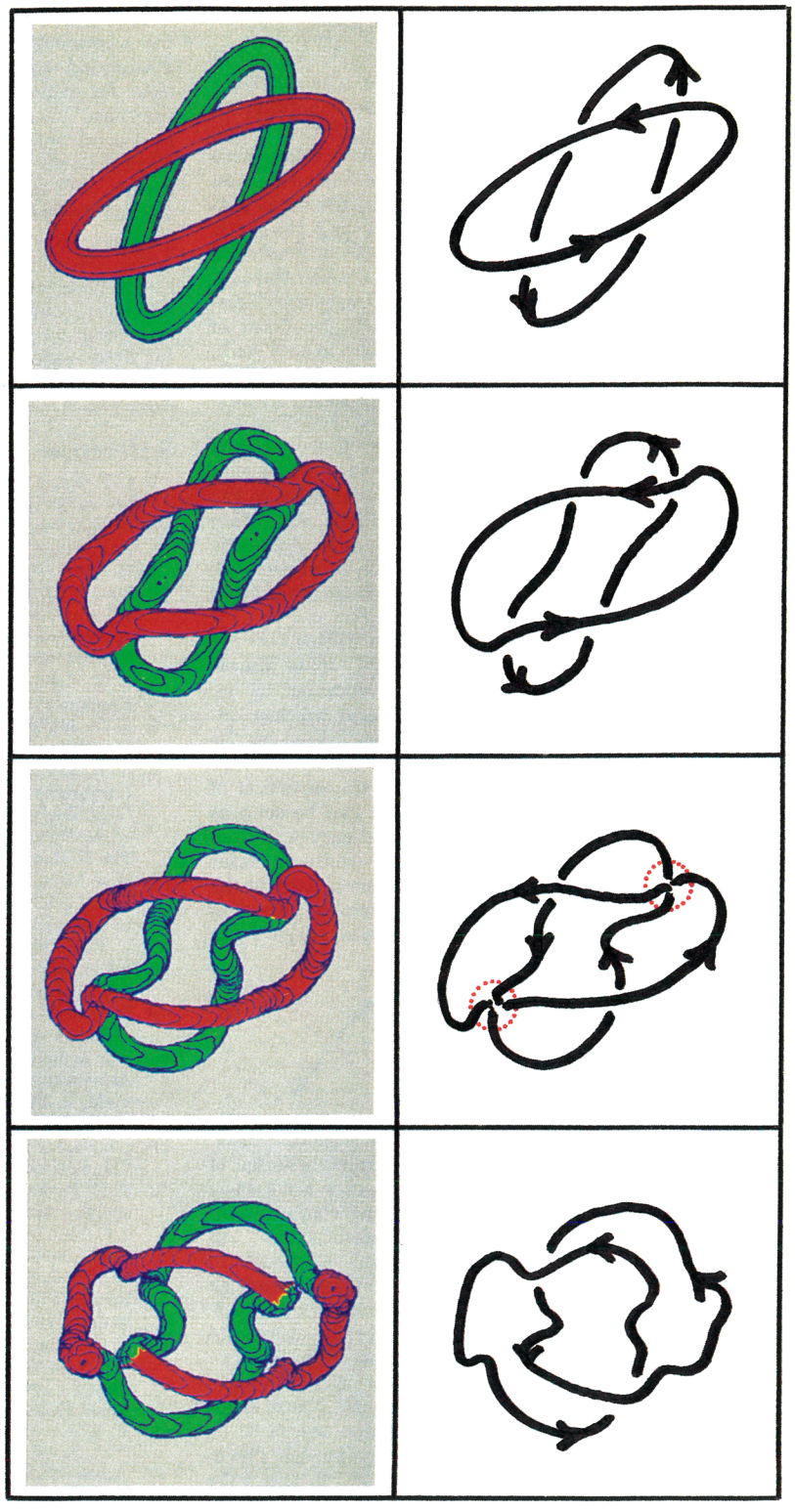
Relaxation of fat knots and charged links

Topologically interesting magnetic equilibria can be found by studying the relaxation of magnetic knots. Start with a knotted magnetic flux tube not in equilibrium. The nonequilibrium Lorentz forces in first approximation induce shortening of magnetic lines. These effects manifest themselves through a tension present in the tube field that makes it behave like a contracting rubber band. Equilibria for magnetic energy can be found by following the physical process of magnetic relaxation using a simple model fluid. A perfectly conducting, incompressible and viscous fluid is a good candidate. Knotted magnetic flux tubes left free to evolve in such a fluid will do so by conserving their magnetic flux Φ and volume V, but converting their magnetic energy into kinetic energy, which in turn dissipates by internal friction. Magnetic links and knots evolve from high to low magnetic energy levels, conserving topology; and because of the induced shortening of field lines under conservation of volume, they become fatter and fatter, with an increase of the average tube cross-section. Evidently, this process of energy reduction must come to a halt when different parts of the tube come in contact with each other: Further relaxation is obstructed by the knottedness and entanglement of the field lines, and a minimum magnetic energy is reached.

Various estimates of magneto-mechanical energy in terms of topological quantities have been put forward in recent years.[6,13,14] These relations give lower bounds



SOFT X-RAY IMAGE OF THE SUN taken by the Yohkoh solar research spacecraft on 12 February 1992. Magnetic fields in the solar atmosphere align the hot x-ray-emitting plasma into filaments. The complexity of the filamentary structure suggests that the magnetic field stores excess energy. When released, this energy can drive flares and eject plasma into space. Topological techniques based on crossing numbers and helicity provide estimates of the amount of energy stored in the magnetic field. The x-ray telescope on Yohkoh was prepared by the Lockheed Palo Alto Research Laboratory, the National Astronomical Observatory of Japan and the University of Tokyo with the support of NASA and the Japanese Institute for Space and Astronautical Science. FIGURE 3



INTERACTION AND LINKING of two elliptical vortex rings. The initial configuration in this numerical simulation is two unlinked rings in parallel planes, placed one in front of the other (top row). As time passes, they interact and reconnect at two sites, with an exchange of strands (second and third rows). The final result (bottom row) is a linked pair of vortex rings with some dissipation and diffusion of vorticity (not visualized). (From H. Aref, I. Zawadzki, reference 18.) FIGURE 4

for the energy levels attainable by knot or link types by taking into account the effects that linking numbers and number of crossings have on the energy of the relaxed state (see equation 3). This means that the least possible amount of magnetic energy that can be attained by the physical knot or link is determined purely by its topology. If topological information sets the levels of minimum energy accessible to the knot or link, geometric properties may also influence the relaxation process. Considerations of helicity and linking numbers, for example, demonstrate that internal rearrangement of magnetic field geometry leads to a spectrum of different asymptotic end-states with the same topology.[10] Moreover, magnetic knots have a natural tendency to get rid of excessive torsion of field lines and S-shaped tube geometries, and this may influence the relaxation process. Perhaps new relations that involve an interplay between geometric and topological quantities will be necessary if we want to understand which equilibria are realizable.

Another case of topological relaxation with different energy functions is given by considering linked electric wires. Two flexible closed wires, such as two rings, linked through one another and carrying static electric charges provide a simple example. Now suppose that the whole system is embedded in a very viscous fluid, like honey, but electrically neutral. The electric charges are confined to the circuits and induce repulsive Coulomb forces that act on the strands of the wires. Because of mutual repulsion, the system progressively relaxes to a least possible energy state by reducing potential and kinetic energy.

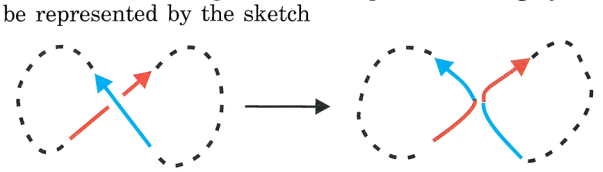
The exact process depends on the actual physical wire: a thin or thick rod, made of a perfect conductor or not. By using various techniques, the accessible energy can again be related to topology via, for example, the minimal number of crossings.[14] The potential energy function is refined further when the elastic properties of the rods are important. Then, elastic tension and internal stress contribute to the relaxation process and modify the system's topological ground-state energy.

Reconnections and change of topology

So far our discussion has explicitly excluded changes of topology. But topological changes do occur when dissipative effects become predominant over the coherency of structures. When this happens there is a dramatic change of fluid patterns, often on small time-scales compared to evolution. The change occurs through the formation and disappearence of physical reconnections in the fluid pattern. In real fluids, for example, vortex and magnetic tubes do interact and reconnect freely. From a dynamical system viewpoint, reconnections take place when the vector field lines (streamlines, vortex lines or magnetic lines) cross each other. If two field lines meet, the point of crossing is a true nodal point, like a bifurcation in a path, where there is more than one choice of direction. Dissipative effects allow the reconnection to proceed through such points.[15]

Analytical and numerical studies of flow patterns show that bifurcations of the field lines occur when configurations are degenerate, as with interfacial flows in the vicinity of a solid boundary. (Think of the flow separation at the nose of an airfoil.) When these events dominate the physics, we can still use a combination of topological, probabilistic and combinatorial techniques to predict average properties and long-term evolution.[16]

As local processes, reconnections are difficult to describe and are still a puzzle for theorists. One simple mathematical approach, which must be mediated by detailed knowledge of the particular physical process, involves techniques of "oriented surgery," performed on the bundle of constitutive vector field lines. Vector field lines are oriented curves, whose arrows give the direction of the field they represent. A physical knot can be seen as a knotted tubular bundle made of oriented curves. When two strands of the bundle come into contact, vector lines of one strand may recombine with vector lines of the other by a "cut and connect" process, which preserves orientation. (See figure 4.) This process of surgery can be represented by the sketch



which shows how a local event may have a global effect.

When this happens, we have a complete change of the topology of the system. Various studies have been done analytically, experimentally and now computationally, to understand the key features of reconnection.[6] An early example of this is given by Kelvin's unsuccessful experiments to produce linked vortex rings, when he noticed how efficiently smoke rings reconnect and reorganize themselves after collision.[2] Recent numerical work, based on direct numerical simulation of the Navier–Stokes equations and of magnetohydrodynamics, attempts to create topologically interesting structures. These studies show that the efficiency of reconnection seems to be strongly influenced by local geometric properties given, for example, by the relative inclination of the tube strands.[17] Orientation-preserving surgery and efficiency of the process are therefore two important features for topological diagnosis of fluid flows.

In dissipative fluids, mathematical and physical properties are no longer conserved, and during the process we lose part of the original information. Some of the invariants, though, are rather robust and may only degrade slowly. One of them is magnetic helicity, the magnetic analogue of the kinetic helicity (equation 1). Its dissipation during reconnection can be modest; in particular, if the reconnection timescale is small compared to classical dissipation times, then helicity loss will be negligible. (See, for example, the paper by Freedman and Berger in reference 16.) The robustness of magnetic helicity plays a central role in fusion plasma physics and in many astrophysical contexts—for example, in the theory concerned with the spontaneous growth of magnetic fields in electrically-conducting fluids. On the other hand, large changes in kinetic helicity are intimately related to qualitative changes in the topology of vortex flows. Helicity and topological estimates, together with detailed knowledge of reconnections, can prove to be very useful for the characterization and classification of the most fundamental fluid mechanisms.[18]

Ricca's work is supported by the Leverhulme Trust.

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