

DISORDER, DYNAMICAL CHAOS AND STRUCTURES

Systems out of thermodynamic equilibrium exhibit particle-like excitations that form planetary systems, dislocations in a lattice of waves or ripples, spatiotemporal disorder and turbulence.

Andrei V. Gaponov-Grekhov and Mikhail I. Rabinovich

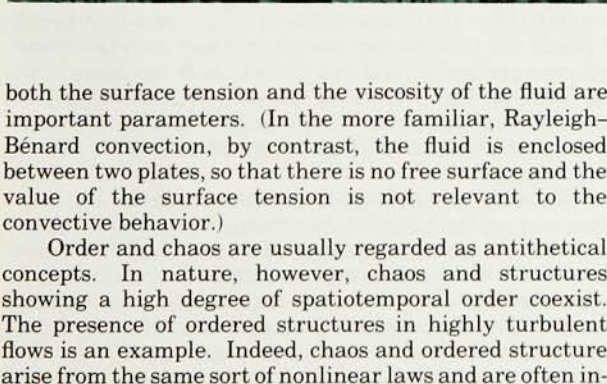
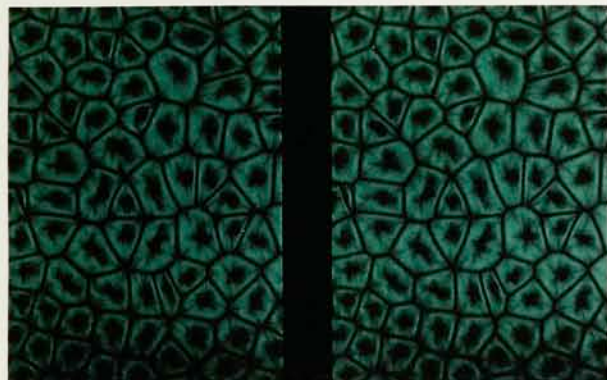
Science, like life itself, often poses questions whose answers are so sophisticated and indefinite that one must either rely on the opinion established by the majority or refer to some article of "faith" rather than deductive knowledge. "Where does randomness come from?" "How does order arise from disorder?" These are perhaps the oldest questions of this kind. Anyone who has thought about the principles governing nature has surely faced such questions. Everyday life, as well as conventional learning in physics, has almost persuaded us that complex, irregular and intricate behavior is possible only in very complex systems and that the study of some of these very complex systems may be beyond the scope of physical theory. The vast multitude of molecules in a bottle filled with gas or a crowd of infuriated fans reacting to the news that a football match has been canceled suddenly, are both examples of complex systems. In such systems we usually fail to relate cause and effect unambiguously—that is, we cannot predict the behavior of the system in detail and therefore conclude that it is random. Of course, there is always the hope that it might be possible to eliminate such seeming randomness and unpredictability if one had more precise knowledge about the system. Would having a detailed knowledge of the interaction among the elements of a complex system and comprehensive and precise information about their initial states allow one to make precise predictions about the system's behavior at large times, so that the system will not appear to be random? The answer is yes only if one believes in Laplacian determinism, which was one of the earliest dogmas of science.

It was believed until recently that the behavior of "simple" systems—that is, systems having a small number of elements (degrees of freedom) and governed by very simple rules—could only be simple and that only complex systems showed irregular or random behavior. Such thinking naturally led to the expectation that stable, regularly organized structures, especially when they emerged in spite of the noise and disorder in the medium, must be caused by some "organizing force" or even by a "builder." Indeed, highly ordered patterns of clouds or hexagonal volcanic formations often appear mysterious and even mystical.

Two remarkable recent discoveries have revolutionized our understanding of randomness and order and overturned centuries-old thinking on their role and origin in nature. Thanks to the efforts of many physicists and mathematicians, it became clear in the late 1960s and early 1970s that simple systems also can give rise to very complex and even random behavior. It seems that randomness of a system's behavior, called dynamical chaos, is inherent to the system, rather than imposed upon it by the environment or other uncontrolled forces: It cannot be removed by a better understanding of any part or aspect of the system.^{1,2} A simple example of such chaos is the disordered, large-amplitude motion of a pendulum under the action of a harmonic driving force.³

At about the same time as the discovery of dynamical chaos there emerged the realization, verified in numerous experiments, that organized regular structures may appear spontaneously from disordered initial states in systems far from thermodynamic equilibrium. As the degree of nonequilibrium increases, the structures evolve into more and more complicated states. One of the earliest and most widely known examples of this behavior is the generation of structures in the form of regular lattices with hexagonal cells at the onset of Bénard-Marangoni convection (see figure 1).⁴ In Bénard-Marangoni convection, a planar layer of a fluid is heated from below, so that

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Evolution of a hexagonal lattice of Bénard-Marangoni convection cells in a horizontal layer of silicone oil heated from below. The rate of heat input was slightly above the critical value for the onset of convection. The cell edges are on the order of a few millimeters long. The pictures, obtained by photographing light reflected from the fluid surface, were taken about 30 sec apart and show the emergence of ordered cells after the onset of convection. (Courtesy of Alexander B. Ezersky.) **Figure 1**

article by Stephen Orszag and Uriel Frisch in **PHYSICS TODAY**, January, page 24.)

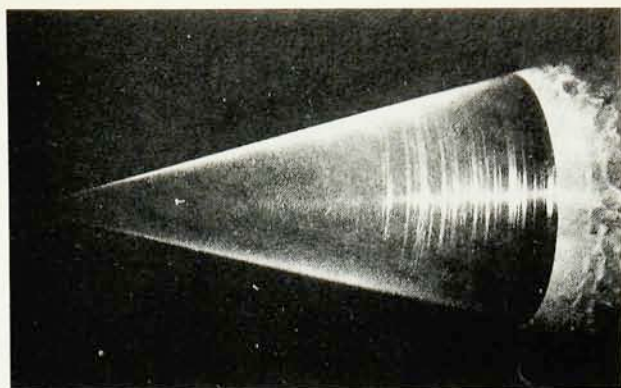
Order in dynamical chaos

Figure 2 shows the lacy structure of a turbulent boundary layer visualized by blowing smoke on the wall of a rotating cone. Both the time dependence of the velocity at a point in the boundary layer (measured, for example, by a hot-wire anemometer) and the distribution of velocity at different points in the layer show no apparent order. Yet the lacy structure is enchantingly beautiful and produces a visceral impression of regularity. There is order in chaos! The degree of this type of order can be determined quantitatively and measured in a variety of processes, such as irregular cardiac beats, turbulent water flows and current pulses in a Josephson junction. The degree of order in chaos is generally expressed in terms of a fractal dimension or entropy.^{5,6}

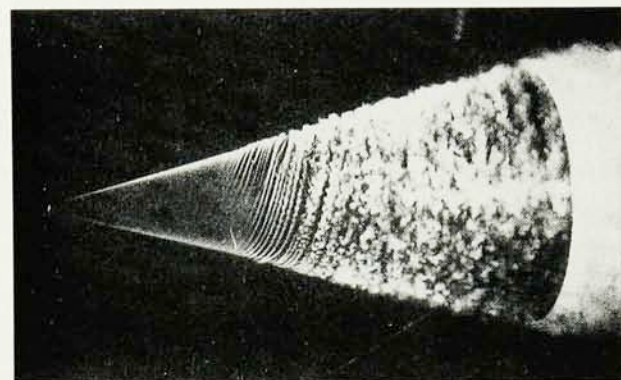
The chaos established in such simple situations as a resonator field, thermal convection or Couette-Taylor flow sets in at a critical value of some control parameter, such as the temperature difference between the plates of a Rayleigh-Bénard cell. At the critical value the behavior of the modes describing the dynamics becomes chaotic. As a rule only a few spatially periodic elementary excitations, similar to harmonic waves, actively participate in such processes. They exchange energy with one another or with a nonequilibrium medium irregularly in time. Thus the chaos in the medium or in the field is temporal. As the control parameter increases beyond the critical value, the number of interacting (linear) modes increases rapidly and the spatial regularity of the field describing the flow is destroyed. A most significant feature of this process is the generation of spatial structures, which are the coherent states of the field—the result of coherent interactions of infinite or many numbers of linear modes. Among these structures may be counted defects in the ordered arrangement of convective cells in a fluid layer, Langmuir solitons

both the surface tension and the viscosity of the fluid are important parameters. (In the more familiar, Rayleigh-Bénard convection, by contrast, the fluid is enclosed between two plates, so that there is no free surface and the value of the surface tension is not relevant to the convective behavior.)

Order and chaos are usually regarded as antithetical concepts. In nature, however, chaos and structures showing a high degree of spatiotemporal order coexist. The presence of ordered structures in highly turbulent flows is an example. Indeed, chaos and ordered structure arise from the same sort of nonlinear laws and are often inseparable. Therefore in most situations it is natural to refer not to the presence or absence of chaos or order but rather to the degree of order or chaos. In this article, we will limit ourselves to a discussion of spatiotemporal structures. (We will not discuss hard turbulence *per se*, which is also sometimes called spatiotemporal chaos, which is not quite exact. For a discussion of hard turbulence, we recommend the excellent, comprehensive



Turbulent boundary layer forms as smoke flows over a rotating cone. The flow develops a lacy structure and further downstream shows an irregular structure of vortices. (From R. Kobayashi, Y. Kohama, M. Kurosawa, *J. Fluid Mech.* **127**, 253, 1983.) **Figure 2**



In the case of multidimensional fields (or media), however, we face an extremely difficult and challenging problem, namely, determining whether equations of classical fields have stable, particle-like solutions. Very few examples of such solutions are known.

Note that for the explanation of some phenomena related to evolution and the birth of structures, it is essential to take into account the noise and fluctuations that can never be eliminated from any experiment. This means that the dynamical equations must be corrected by the addition of terms that describe the sources of the noise and fluctuations relevant to the experiment. Systems whose behavior is in some way affected by noise are called stochastic. Studies of the spatiotemporal behavior of the localized structures of stochastic nonlinear fields present several interesting problems. Localized defects on interfaces or surfaces, vortices in superfluid helium, and large coherent structures in turbulent flows are all examples of localized structures in stochastic systems. We will not discuss stochastic nonlinear systems in this article; the interested reader may consult references 7 and 8.

Localized states of nonlinear fields

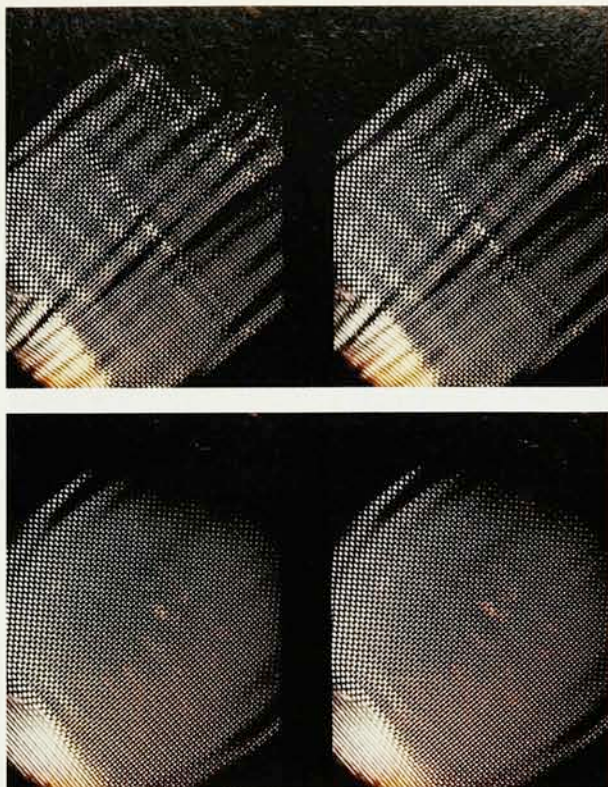
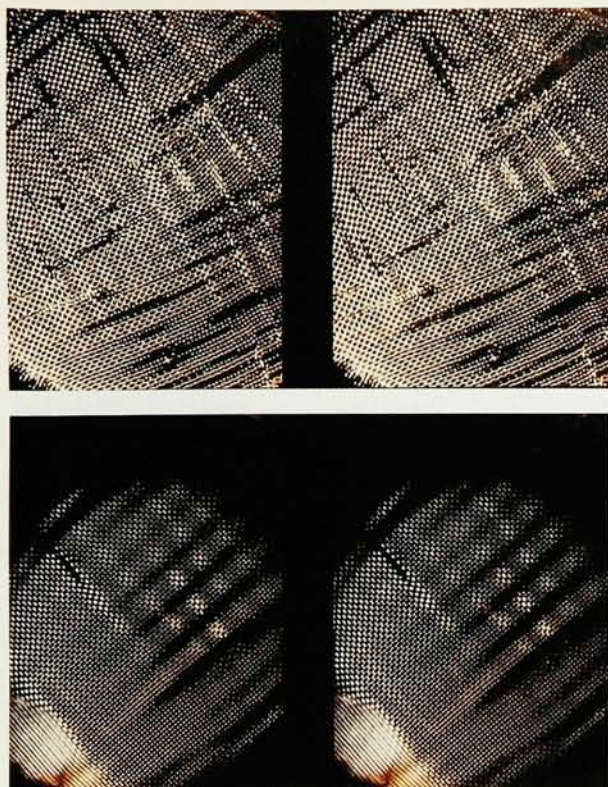
Stable, localized states of nonlinear fields are of great interest in various branches of physics and astrophysics. Einstein was the first to propose that elementary particles may be regarded as localized states of nonlinear fields. Heisenberg had a similar point of view. The localized structures of gravitating gas clouds are associated with galaxies, clusters of galaxies, and other formations. Localized charge density waves are interesting for superconductivity. The large eddies of cold water that break through the Gulf Stream boundary layer into the lower Atlantic are stable localized structures in the ocean, and the blocking highs (blocking events—anticyclones in the atmosphere), which give Europe wonderful summers, such as those of 1976 and 1989, are examples of such structures in the atmosphere.

It should be emphasized that the localized states may set a size on the background structured medium. These are envelope structure defects. Figure 3 shows one such structure.⁹ When a fluid layer is subjected to vibrations in a gravitational field, its free surface breaks into Faraday ripples. (The phenomenon is named after Michael Faraday, who described it over 160 years ago.¹⁰) As the sequence in figure 3 shows, an absolutely regular lattice of square cells exists for a range of values of the amplitude of vibration, but several complicated structures and defects appear in the lattice as the amplitude increases beyond that range. In general, defects in periodic lattices are examples of localized structures. They move, usually chaotically, and interact with one another. The chaotic spatiotemporal dynamics of defects can be regarded as an aspect of turbulence.¹¹

in a plasma, and coherent vortices in a shear flow. Such structures are localized in space and may have their own temporal dynamics. They may form bound states or walk randomly in space. They may be stable or metastable. The only important feature they must have—the one to which they owe their existence—is that they must be long-lived on the time scale of the elementary modes of the system.

An important factor that allows us to pass over from the description of continuous fields (velocity temperature and others) to the analysis of the dynamics of discrete elements (structures) is the small value of the parameter μ , which is the ratio of the value of the field of the j th structure at the center of mass of the i th structure to the value of the field of the i th structure at its own center of mass. The smallness of this parameter suggests that the structures are not very dense or that the interactions between them are small.

The theory of ordered structures starts from the assumptions that there exist localized states of nonlinear fields and that those states are stable and robust. A one-dimensional soliton is an example of such a stable structure.



Spatiotemporal structures in Faraday ripples. Capillary waves were excited in a 10-mm-deep layer of silicone by oscillating it vertically (in a gravitational field) at 140 Hz. A periodic modulation appeared in the background lattice of the capillary waves. Increasingly complicated spatiotemporal structures appear with further increases in the amplitude. (Courtesy of A. B. Ezersky) **Figure 3**

The defects not only move and interact, they also collapse and split. Their number may change with time, but suitably defined time averages of the number of defects depend only on the degree of nonequilibrium of the medium. The number of defects in the background of Faraday ripples increases with the pumping amplitude, as shown in figure 3. In fluids, the number of defects increases with increasing Reynolds number. This result about the number of defects appears to be quite general. In particular, L. Fil, G. Lega and G.L. Meunier studied,¹² in a numerical experiment on a Cray-2 supercomputer, fluctuations in the number of defects in the framework of a two-dimensional Ginzburg-Landau equation with complex coefficients (similar to equation (11)). They found that not only are the trajectories of the defects (or localized structures) random, but so is their number! And all these features arise in a purely dynamical system, one not subjected to any fluctuations or external noise.

The laws of defect interaction are quite universal. The topology of defects and the dynamics of their appearance or disappearance are very much the same for Faraday ripples and for structures seen when convection is induced by a thermal gradient in a fluid or by electric forces in liquid crystals (see for example reference 13).

When constructing a theory of spatiotemporal structures, it is natural to distinguish between two radically different situations: In the first case, the localized states of a nonlinear field interact weakly, retaining their individuality; in the second, the interaction is strong, so that a region may alternate between being normal and being structured (that is, structures appear and disap-

pear). We are interested here primarily in the spatiotemporal dynamics of weakly interacting structures. A rather effective, albeit only asymptotic (and hence approximate), theory can be constructed for the description of such structures. The theory allows the very attractive possibility of passing from continuous models (involving partial nonlinear differential equations) to discrete equations for localized structures or their ensembles.

Localized structures in conservative and dissipative fields may be considered in parallel for a significant class of nonlinear fields, namely, potential fields. It is these fields that we shall be concerned with in this section.

In the 1970s remarkable advances in studies of soliton solutions in one-dimensional theories aroused enthusiasm and stimulated active search for solitons and dissipative localized structures in two- and three-dimensional theories as well. It appeared, however, that two- and three-dimensional soliton solutions in nonlinear field models are, as a rule, unstable. For example, for some field variable $u(\mathbf{x}, t)$ whose dynamics are given by the model equation

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u + f(u) = 0 \quad (1)$$

(where f is some potential function), the localized states, according to the so-called Derrick-Hobbarth theorem, spread out in two and three dimensions. How can this model be modified so that the "particles" (localized states) become stable?

One of the reasons for the lack of stability of the localized states of equation 1 is that the model has no

universal spatial scale. The characteristic scale of particles is determined by the magnitude of the nonlinear term in f , usually a quadratic term, which also sets the scale for the particle mass. The spatial dispersion (or the existence of an intrinsic, independent length scale in the medium or field) is likely to lead to the "universalization" of the parameters of static particles and thereby prevent their spreading. A necessary condition for particle stability is that the minimum of the potential energy density occurs at the centers of localized states of interest.

Thus the search for stable structures is the determination, based on symmetry requirements, of different types of invariances of the localized field states that would minimize the potential energy or the free energy F .

We shall restrict our consideration to very simple models of scalar and isotropic fields. The energy density $\mathcal{F}$ ($F = \int \mathcal{F} d\mathbf{r}$) can be represented in the form of a power series in the field and its gradient close to a trivial homogeneous state (for isotropic fields):

$$\mathcal{F}_1 = au^2 + \beta u^3 + \gamma u^4 + \xi(\nabla u)^2 + \zeta(\nabla^2 u)^2 + \dots \quad (2)$$

Similar series expansions can be written for complex scalar fields, except that the powers of the field are replaced by the powers of the field modulus, so that the free energy remains real. The terms proportional to ∇^2 take into account spatial dispersion of the field. We can obtain an equation for the field distribution realized in a static structure by setting the variation of F relative to u (or u^* in case of complex fields) equal to zero:

$$\frac{\delta F}{\delta u^*} = 0 \quad (3)$$

It is instructive to recall here the simple analogy of a ball moving down the rough bottom of a bath. When the bath is filled with a highly viscous fluid the ball, moving under the potential force (gravity) and the frictional force,

slides down gradually and comes to a stop in one of the nearest pits. This process is described by a gradient equation:

$$\frac{du}{dt} = -\frac{\delta F}{\delta u} \quad (4)$$

When by contrast, the ball moves down the same rough surface, because of the same potential force but in the absence of any viscous liquid, it does not sit still at the bottom of the pit but oscillates about it. Next, if we place the ball at the bottom of the pit and do not impart to it any (initial) velocity, it stays in that position forever. In the absence of viscosity the motion of the ball is described by a second-order equation of the form

$$\frac{d^2 u}{dt^2} = -\frac{\delta F}{\delta u} \quad (5)$$

Clearly the stable equilibrium in systems (4) and (5), corresponding to the minimum of F , is identical regardless of the presence of the viscous fluid. The only difference is that the dissipative system finds these minima by itself for arbitrary initial conditions, while in the purely potential case (equation 5), one must guess or calculate the minima by some means and then leave the system in one of them.

Our analogy, however good or interesting, turns out to be not so complete when we consider the stability of particle-like states of nonlinear fields. A solid ball has no intrinsic degrees of freedom, and therefore the necessary condition for the stability of a potential energy minimum is at the same time a sufficient condition. The situation is different with respect to localized field structures. Such structures typically have intrinsic excitations, which, as a result of nonlinear interactions, may remove energy from the ground state and make it unstable.

When we supplement equation 3 with a term similar to that in equation 5 to describe the field "inertia," we obtain a model equation of the form

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\delta F}{\delta u} = -(k_0^2 + \nabla^2)^2 u - u(1 - \beta u + u^2) \quad (6)$$

for a real scalar field. (We can similarly add inertia terms to the equivalent of equation 2 for complex fields.) The parameter k_0 defines a universal spatial scale typical of all inhomogeneous static states of the field u .

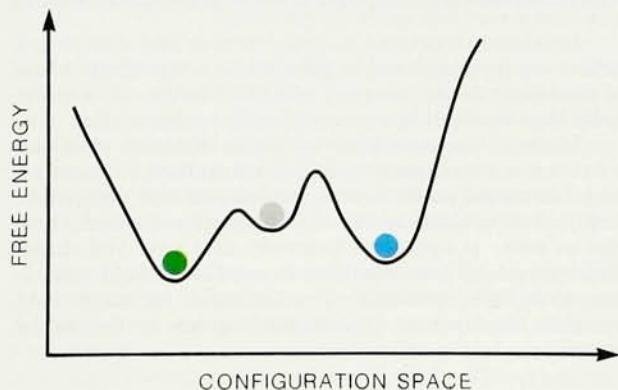
The similarity of the model equation 6 to equation 5 means that the polynomial terms in the field u or its derivatives may be obtained by minimizing some potential (or free-energy functional). The model defined by equation 6 can be related to the dissipative models, defined by

$$\frac{\partial u}{\partial t} = -\frac{\delta F}{\delta u} \quad (7)$$

The value of the free energy may only decrease in time in all solutions of equation 7. This is because

$$\frac{dF}{dt} = - \int \left| \frac{\partial u}{\partial t} \right|^2 d\mathbf{r} < 0 \quad (8)$$

Thus, as $t \rightarrow \infty$ the solutions of equation 7 must converge only to stable equilibrium states corresponding to the minimum of the free energy, just as in our analogy of the ball moving down a bath filled with a viscous fluid. One



Schematic free energy functional of a nonlinear field $u(\mathbf{r}, t)$. Three types of minima are indicated: A minimum in which the value of the field is 0 at all points in space (green), one in which the field is nonzero at some localized set of points (gray) and one in which the field has a nonzero constant value everywhere (blue). **Figure 4**

says that the minima of the free energy are static attractors of the dynamics in phase space. There may be a great number of equilibrium states, a case usually referred to as multistability, and the corresponding spatial field distributions may be either regular, as in the Bénard cells, or almost regular with several defects, as in the Faraday ripples at large amplitude shown in figure 3.

For a dissipative medium modeled by an equation of the type defined by equation 7, the possibility of localized structures arises from the coexistence of more than one homogeneous stable state for a range of values of the control parameter. This is illustrated in figure 4. As discussed above, every homogeneous stable state of the medium corresponds to a minimum of the free energy. One such minimum occurs at $u=0$. It is natural to assume that under suitable conditions the free-energy functional may have other minima that correspond to spatial configurations that are localized in space. In a system possessing spatial dispersion, described in equation 6 by the ∇^4 term, the minima corresponding to localized structures are indeed stable, as indicated in figure 5.

Our goal is to find stable particle-like states among the equilibrium states. Apparently the field configurations in nondissipative models such as those described by equation 6 depend critically on initial conditions, and it is impossible to guess the field distributions corresponding to stable particles. The gradient models given by equation 7, whose equilibrium states are identical to those of equation 6, may be of use in this context, because the dissipative system itself converges to the equilibrium states.

In a two-dimensional geometry, stable localized states of a nonlinear field described by an equation similar to equation 7 are shaped like disks; the field oscillates, the amplitude of the oscillation decreasing exponentially with the distance from the core so that it becomes zero at infinity. Figure 5 shows localized states of a real field found in three dimensions in a dissipative model.

Localized states exist also for complex fields. These states are shaped like spiral waves in two dimensions and toroidal scrolls in three dimensions. Complex fields also exhibit localized structures that are long-lived but metastable.

Bound states and chaos

In contrast to the power-law decay of the fields of classical solitons, the fields of stable localized structures—obtained by solving equations of the type 6, for example—decay exponentially with the distance from the point of the field maximum, in an oscillatory fashion. The oscillations are evidently one of the manifestations of the spatial dispersion of the fields, which, as noted earlier, is needed to obtain stable localized structures.

Because of the exponential decay with distance, the fields at the periphery of a localized structure are weak, so that the fields of two structures overlap only slightly when they are far apart and the interaction between them is weak. As a result, the parameter μ , defined earlier, is small. We may exploit this fact when trying to describe the dynamics of localized structures and to pass from the continuous models to equations of motion for the mass centers of the structures, which from now on we shall call particles. These equations are similar to the equations of Newtonian mechanics. In particular, for centrosymme-



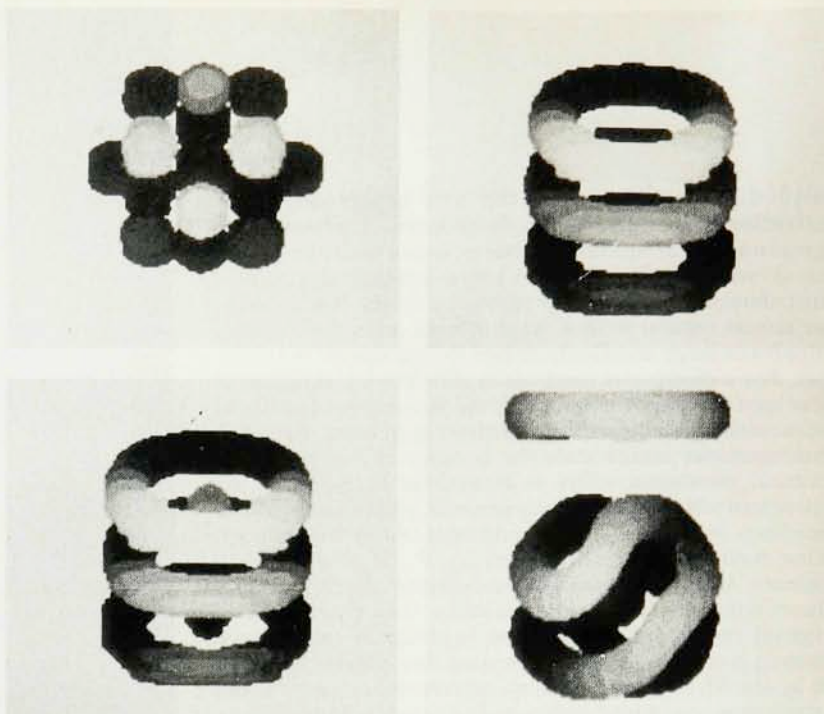
Spatially localized configurations of a three-dimensional nonlinear field: a "ball," a "torus" and a "baseball." (Adapted from ref. 14) **Figure 5**

tric localized structures in a dissipative model, we obtain the system of equations¹⁴

$$\frac{d\mathbf{r}_{0i}}{dt} = \nabla_{\mathbf{r}_{0i}} \sum_{j \neq i} \text{Re} \frac{e^{ik|\mathbf{r}_{0i} - \mathbf{r}_{0j}|}}{|\mathbf{r}_{0i} - \mathbf{r}_{0j}|} = f_i(|\mathbf{r}_{0i} - \mathbf{r}_{0j}|) \quad (9)$$

Here $\mathbf{r}_{0i}$ is the coordinate of the i th particle and $f_i(|\mathbf{r}_{0i} - \mathbf{r}_{0j}|)$ is the force, which depends on the spatial distribution of the field in the "tails" of the structures.

When we pass from the continuous equations for the fields to discrete equations for their particle-like states, we must solve a problem that is in a certain sense the converse of the traditional one in elementary-particle physics. In our problem, we must separate out particles from the field equations and determine the forces that



Bound states of stable spatially localized states, or "elementary" particles, in a dissipative model of a nonlinear field. All of the localized states shown in figure 5 form bound states. (Adapted from ref. 14.) **Figure 6**

govern the interaction of the particles in one another's potential fields. This procedure, intricate as it is, has its rewards: Instead of the rather cumbersome nonlinear field equations, we can use the much simpler and more conventional equations of mechanics.

Although the particles obtained within the Hamiltonian and the corresponding dissipative models are identical, the dynamics of potential and dissipative particles are qualitatively different, just as they were in the analogy we discussed. The particles described by dissipative models move spontaneously, come to a stop and form bound states of clusters (see figure 6). Because of the oscillatory tails of individual particles, the bound states may include an arbitrary number of localized structures. The bound states correspond to equilibrium in the system given by equation 9 and have various spatial configurations: They may be regular or irregular chains of structures, lattices (in particular, quasiperiodic lattices), clusters with defects of different types and so on.

The structures of the Hamiltonian fields behave in a different fashion. In particular, for the field described by equation 5 instead of equation 9 we have

$$\begin{aligned} \frac{d\mathbf{r}_{0i}}{dt} &= \mathbf{V}_i \\ \frac{d\mathbf{V}_i}{dt} &= f_i(|\mathbf{r}_{0i} - \mathbf{r}_{0j}|). \end{aligned} \quad (10)$$

The equilibrium states here are of course the same as in equation 9, but they are no longer attractors. Instead, the structures may rotate and vibrate about configurations close to those of the equilibrium states. Like the equations of the Hamiltonian mechanics, the set of equations 10 describes, in particular, the planetary rotation of the particles relative to each other. The difference between this new, fascinating world and the Newtonian one lies in the interaction potential between objects. In this world the particles may attract or repel one another. There may also exist infinitely many orbits, corresponding to the infinitely many minima in the interaction potential

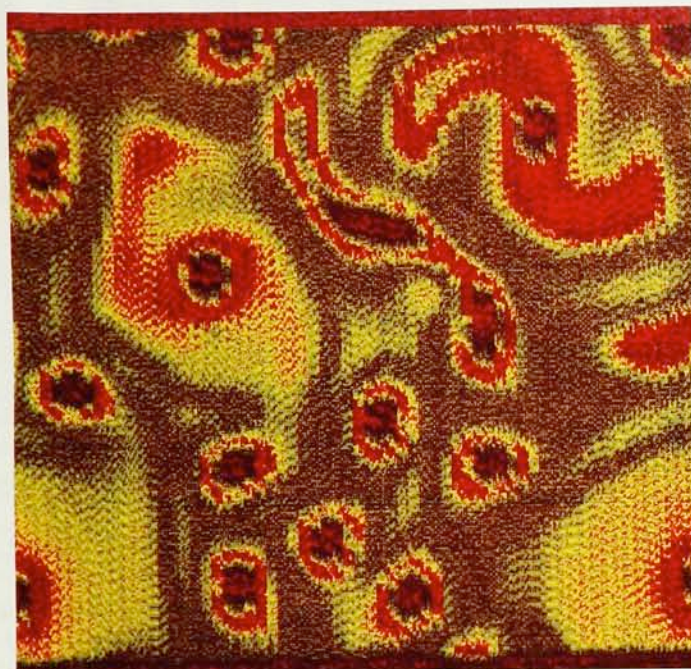
between structures. Here we can draw analogies with the discrete levels of quantum systems, but that discussion is beyond the scope of this article. Let us, however, remind ourselves that our particles arise from complicated field configurations, and let us pause to imagine that the dynamics of these particles can indeed cause mixing and randomness in the medium.

Stochastic dynamics of ensembles of structures

Direct computer experiments on a nonlinear complex field described by an equation similar to the complex counterpart of equation 6 have confirmed the results briefly mentioned above about the asymptotic behavior of the dynamics of localized structures. Similar results were obtained earlier for other models.¹⁵

Computer visualization of the random motion of particles and the accompanying changes it causes in the medium convinces even inveterate skeptics that the origin of spatiotemporal disorder in nonlinear media (or fields) can be understood in the context of simple models for the dynamics of the medium.

Our particle-like structures have intrinsic degrees of freedom. Excitations in the boundary-layer vortices in figure 2 or amplitude and phase oscillations in the capillary cell in figure 3 are examples of the intrinsic degrees of freedom. As we saw in figure 3, the structures themselves may show spatiotemporal disorder. It is therefore useful to study ensembles of structures. If the structures form a regular lattice, for example, the spatiotemporal disorder is a random walk of excitations that represent the change in the states of the elements in the lattice. When constructing a theory of the chaos in spatiotemporal structures, it is convenient to employ amplitude equations that describe, for example, the dynamics of the amplitude of a complex field in time and space. This class of equations includes the equation describing modulation waves on a background of capillary ripples,⁷ as well as the potential models described by equations 6 and 7, which are generalizations of the well-known Ginzburg-Landau model.



Solution of equation 11 for a lattice of 64×64 points with $\beta = c = \sqrt{3}$ and $\kappa \sim 5$. Color indicates modulus of the amplitude of the complex field. (Adapted from ref. 15.) **Figure 7**

Consider as an illustration the processes described by a lattice analog of a traditional Ginzburg-Landau equation:¹⁶

$$\frac{da_{lj}}{dt} = a_{lj} - (1 + i\beta)|a_{lj}|^2 a_{lj} + \kappa(1 - i\epsilon)(a_{l,j+1} + a_{l,j-1} + a_{l+1,j} + a_{l-1,j} - 4a_{lj}) \quad (11)$$

where $l, j = 1, 2, \dots, N$ and $a_{lj}(t) \equiv a_{l+N,j+N}(t)$ is a complex amplitude for the lj th element of the lattice. This model permits investigation of the structure of spatiotemporal chaos (see figure 7). Calculations show that new elementary excitations appear in the system as $1/\kappa$, which measures the degree of nonequilibrium in the system, increases. For small values of $1/\kappa$, the interaction between the excitations gives rise to regular modulation lattices, which are described by superpositions of mutually orthogonal standing waves. But as $1/\kappa$ grows, spiral modulation waves appear whose field intensity is zero at the center and rapidly acquires a constant value at the periphery. Next the number of structures—spirals, two-dimensional vortices and so on—grows, and they begin to move randomly over the lattice, producing chaos.

If the chaos is assumed to be spatially homogeneous (when suitably averaged over time), it can be described quantitatively using the concepts of the Kolmogorov-Sinai entropy,^{1,5,16} which characterizes the degree of dynamical disorder, and the fractal dimension.^{5,16} The fractal dimension is, in some sense, an estimate of the number of effective degrees of freedom that take part in the onset of chaos. It should be noted that the fractal dimension in chaos is usually much lower than the total number of excited linear modes. This again proves that many nonlinear modes are synchronized and form clusters. (The clusters correspond to nonlinear structures in the spatiotemporal representation.) It is the number of individual clusters corresponding to new "nonlinear modes" that determines the fractal dimension.

Self-generation of chaos downstream

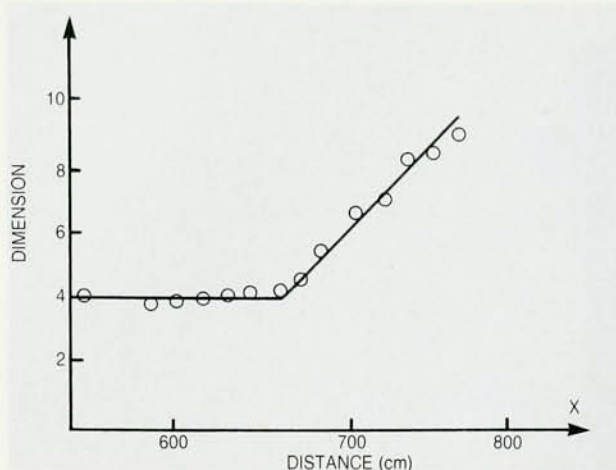
We now turn to shear flows, to examine whether the concept of chaos in spatiotemporal structures is useful for the description of turbulence in fluid flows. Hydrodynamical experiments, with their remarkable ability to be visualized (see figure 2), encourage the construction of models involving spatiotemporal structures for describing turbulence.

In many shear flows a regular flow is converted to a turbulent one after a series of transformations along the flow. These so-called spatial bifurcations correspond to the birth and amplification downstream of new excitations or degrees of freedom of the flow. The spectrum of these excitations is essentially continuous. An analytic description of such a process is beyond our current capabilities. It is understood, however, that the nonlinear evolution of the primary instability in the flow generally leads to the formation of structures similar to the vortices in figure 2. As a result, the subsequent evolution of the flow can be considered as the transformation of excitations in the chain of structures as those excitations act on one another consecutively.

A phenomenological model describing the nonlinear dynamics of a flow having structures, labeled by j and having transverse coordinates y , arranged in one direction is given by¹⁷

$$\frac{\partial u_j}{\partial t} - \phi(u_j, \lambda) + \eta \frac{\partial^2 u_j}{\partial y^2} = \gamma(u_j - u_{j-1}) + \kappa(u_{j+1} - 2u_j + u_{j-1}) \quad (12)$$

where $j = 1, 2, \dots$. Here γ is a measure of the interaction between neighboring structures, κ is a diffusion coefficient and ϕ is some (polynomial) function. For flow systems $\gamma \gg \kappa$. The left-hand side of the equation describes the intrinsic spatiotemporal dynamics of an individual structure while the right-hand side describes the interaction between the structures. In most cases, the feedback "upstream," given by the term proportional to κ , can be



Fractal dimension at points in a boundary layer of a flow excited by a vibrating ribbon in a wind tunnel. The fractal dimension was obtained from the time series for the velocity, which was measured by a hot-wire anemometer. The increase in fractal dimension downstream from the vibrating ribbon indicates the increasing complexity of the flow. (Adapted from ref. 18.) **Figure 8**

considered as the perturbation. Equation 12 must be supplemented with boundary conditions. In the discussion below we shall assume that $u_0(t) = Ae^{i\omega t}$, which corresponds to the time-periodic excitation of the flow.

When diffusion is neglected ($\kappa = 0$), the problem of the origin of spatiotemporal chaos along the flow reduces to determining whether a strange attractor appears in the phase space of the chain of elements. A flow having discrete structural elements typically has spatially inhomogeneous steady-state solutions. When such solutions are stable, chaos does not set in downstream and the flow continues to be laminar. If for some j^* the solution becomes unstable, then for $j > j^*$ a more complicated dynamics is established that generally has a higher dimension. The new dynamics which may be, quasiperiodic, for example, may itself become unstable for some j^{**} , and this process may continue, with increasingly more complicated dynamics appearing downstream, until chaos sets in at some j_{cr} . Strictly speaking, this means that in the phase space of the system comprising $j > j_{cr}$ dynamic elements, there exists (under the specified boundary conditions) a stochastic limiting set—a strange attractor.

By changing the dynamical properties of the point elements of the medium, we can describe different scenarios for the spatial evolution of turbulence—through period doubling, quasiperiodicity, intermittency and so on.

In our theory the evolution of chaos downstream corresponds also to the growth of the fractal dimension of the time series (for some variable, such as the velocity) along the coordinates. This is shown in figure 8 for the fractal dimension of a real boundary layer near a plate.

Does the fractal dimension continue to increase as one moves downstream? How is it related to the Reynolds number? To the number of structures in a turbulent flow? Answers to these and other questions in the theory of turbulence must await further studies (see, for example, reference 19).

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