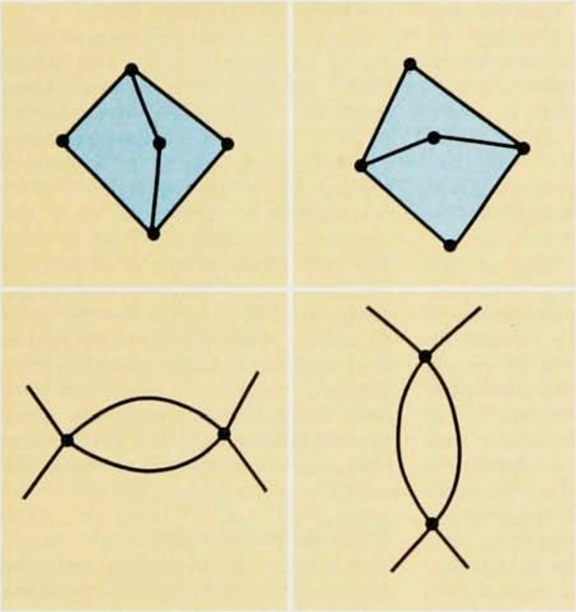
MODELS OF STRING THEORY AND 2-D QUANTUM GRAVITY ARE SOLVED EXACTLY

In how many ways may a quadrilateral be dissected into a *given* number of smaller quadrilaterals? Or a hexagon into triangles? Or, for that matter, any polygon into given numbers of other polygons? Such questions about enumeration of dissections of a planar surface attracted the attention of mathematicians—most notably, of Leonhard Euler—in the middle of the 18th century. Mathematicians solved many cases of such problems using methods of graph theory in the 1960s. In 1978, however, physicists, in their attempts to enumerate Feynman diagrams in certain theories, independently developed a rather elegant and general method that can be used to enumerate dissections of any surface. Physical insights into the divergences in the solutions obtained by those general methods have now led to exact solutions—the first nontrivial exact solutions to date—of simple models of quantum gravity in two dimensions and of simple string theories. String theorists hope that the solutions will provide much-needed insights into the structure of those theories, just as solving the Schrödinger equation for simple, one-dimensional potentials yields insights into quantum dynamics: For example, the solutions for one-dimensional potentials show features, such as the existence of the zero-point energy and the possibility of tunneling, that distinguish quantum mechanical behavior from the classical one. In the absence of any exact solutions, attempts to understand the content of string theories had been limited to perturbation expansions in the string coupling constant. The new, exact solutions show that string theories have features, not fully understood yet, that no perturbation expansion in the coupling constant could unravel.



Dissecting a quadrilateral into two quadrilaterals in two ways (top) gives two Feynman diagrams (bottom) as duals of those dissections. In general, duals to dissections of an n-gon are Feynman diagrams with n external lines, and duals to dissections of an n-gon into m-gons are Feynman diagrams for a theory with mth-order coupling. The correspondence, shown here for a plane, holds for surfaces of all genera.

The perturbation expansion in the string coupling constant is similar in spirit to the perturbation theory used in elementary quantum mechanics. When such expansions are used in field theory, the number of terms at a given order in the expansion parameter often increases much faster than any power of the order, so the expansion diverges. The divergence of the expansion, especially when the signs of successive terms in it do not alternate, is usually evidence for nonperturbative effects, such as tunneling. The divergence may also indicate a poor choice of the vacuum state (in quantum mechanics parlance, of the unperturbed wavefunction). In string theories, the perturbation expansion has been known to be divergent. And the vacuum state, which in string theories determines such important issues as the particle spectrum and the gauge groups, often has too much symmetry. But perturbation expansions yield no mechanism for breaking some of the unwanted symmetries. Supersymmetry—the symmetry between fermionic and bosonic degrees of freedom in a theory, which superstring theories have—is an example of a symmetry that must be broken. For these reasons, David Gross (Princeton) explained to us, the lack of nonperturbative methods in string theory has been a stumbling block in the further development of the theory.

The universe we live in may be one of many possible universes. Quantum gravity theories may permit "communication" between different universes. The physics of wormholes, which are space–time configurations connecting different possible universes, may be grasped, Stephen Shenker (Rutgers University) told us, only in a nonperturbative treatment

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of quantum gravity. Wormholes and other nonperturbative aspects of quantum gravity may also be necessary for understanding why the cosmological constant vanishes (see PHYSICS TODAY, March 1989, page 21).

Genealogy

The exact solutions of models of string theory and quantum gravity now being discussed are a culmination of—or, regarded more hopefully, just the most recent link in—a long progression of ideas. The progression dates from the mid-1970s, when theorists were eagerly looking for simple models and approximations that might reveal the physics of quantum chromodynamics, which had just emerged as the successful theory of strong interactions. In that climate Gerard 't Hooft (University of Utrecht) proposed, in 1974, that when the symmetry group of QCD (which is SU(3) because quarks come in three colors) is generalized to SU(N), $1/N$ emerges as an extremely useful expansion parameter for studying the theory. 't Hooft demonstrated the usefulness of this expansion parameter by arguing that many of the salient features of strong interactions occur already in the leading order in the $1/N$ expansion. The leading order, obtained in the limit $N \to \infty$, is called the planar approximation (for reasons to be discussed later). (For a discussion of $1/N$ expansions and their relevance to hadron dynamics, see the article by Edward Witten, in PHYSICS TODAY, July 1980, page 38.)

't Hooft's hopeful suggestion and the desire to master the planar approximation that it engendered among theorists led to the study of simple models in which, as in the SU(N) generalization of QCD, there is an $N \times N$ matrix at every space–time point. In 1978, in what has turned out to be a remarkable development in mathematical physics, Edouard Brézin (Ecole Normale Supérieure, Paris), Claude Itzykson (CEN Saclay), Giorgio Parisi (University of Rome) and Jean Bernard Zuber (CEN Saclay) solved a class of matrix models in the planar limit for the simple case in which the space–time consists of a single point. That solution provides, as mentioned above, a convenient way to enumerate the Feynman diagrams of the theory as well as possible ways to dissect polygons. But the solution is regarded as a remarkable development because of the new ideas the four authors, as well as Daniel Bessis, Sudhir Chadha, Gilbert Mahoux and, especially, Madan Lal Mehta (all at CEN Saclay), introduced in subsequent papers rederiving and generalizing the solution.

In 1985, Jan Ambjorn, Bersign Durhuus and Jürg Frohlich (ETH Zurich), François David (CEN Saclay) and Vladimir Kazakov (Cybernetics Council, USSR Academy of Sciences, Moscow, and ENS Paris) proposed that the sort of matrix model solved in 1978 might be used to study string theory and two-dimensional quantum gravity. This correspondence was confirmed three years later, in 1988, when the late Vadim Knizhnik, Alexander Polyakov (Princeton) and Alexander Zamolodchikov (Princeton) obtained an exact solution for a special case of two-dimensional gravity—one in which the space–time is limited to a spherical shape. That solution agreed with the one obtained earlier, in 1985, using the matrix-model ideas.

The recent, exact solutions of models of quantum gravity and string theory emerged from the realization that matrix models like the one solved in 1978 in the planar approximation can be solved exactly in a certain scaling limit (discussed later) that corresponds to special values of the coupling constants of those models. The first solutions appeared last fall in three related but independent papers—by Brézin and Kazakov, by Shenker and Michael Douglas (Rutgers) and by Gross and Alexander Migdal (Princeton).[1] Those first solutions have been extended to more general situations by the authors and by a number of theorists around the world.

Sums over surfaces

The motion of a particle may be described by a catalog of its positions or some generalized coordinates as a function of time. In classical mechanics the particle, in its motion between two fixed points, follows the trajectory along which the action is minimum. In quantum mechanics as formulated using Feynman path-integral methods, by contrast, the amplitude for the particle's making a transition between the two points is given by a sum over all possible trajectories between the two points, each trajectory being weighted by an exponential of i $(=\sqrt{-1})$ times the action along that trajectory. Similarly, the quantum mechanical description of the dynamics of a string involves a sum over all possible surfaces connecting the string's initial and final configurations. The sum depends on the model chosen for the string dynamics because the contribution of each surface depends on the value of the action on that surface. The sum also depends on the dimensionality of space–time because the value of the action depends on that dimensionality and, more importantly, because the shapes a two-dimensional surface may take depend on the dimensionality of the space—in this case, the space–time dimensionality—in which the surface is embedded.

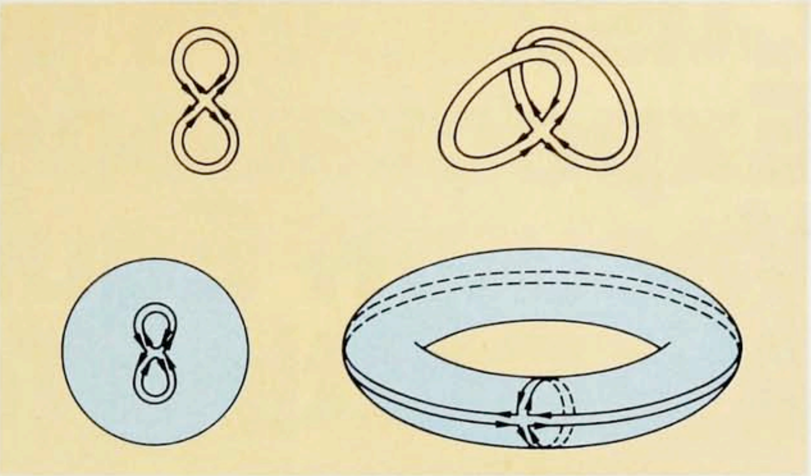
The quantum mechanical description of gravity in two-dimensional space–time also involves a sum over two-dimensional surfaces: Like the principle of least action of classical particle dynamics, the Einstein equations of general relativity, whose solution for a given set of boundary conditions specifies the shape or topology of space–time, provide only a classical description of gravity. In general, quantum theories of gravity involve sums over objects of dimensionality equal to that of space–time. Thus the fundamental mathematical problem underlying studies both of quantum gravity in two space–time dimensions and of string theories is the enumeration of surfaces of different shapes and sizes and the summation of their contributions to some suitably defined Feynman path integral.

A curve with free ends that does not intersect itself can be smoothly transformed into a straight line and is said to be topologically equivalent to it. In general, a one-dimensional curve may be characterized by the number of times it intersects itself. Such a characterization of curves, and therefore of particle trajectories, is the reason that the perturbative, Feynman-diagram expansions of theories of point particles are also called loop expansions. Similarly, every surface with no free boundaries is topologically equivalent to a sphere that has a few handles attached to it. Such a surface is said to have genus p if it may be smoothly deformed into—that is, is topologically equivalent to—a sphere having $2p$ holes joined in pairs by p handles. The perturbative expansions in the string coupling constant, which hitherto have been used to study string theories, are arranged according to the genus of the surface spanned by the string. Such expansions, however, are badly divergent.

The recent exact solutions simultaneously sum the contributions from surfaces of different genera. In the method used to achieve this summation the continuous surface is first approximated by polygonal tiles. The continuum limit is recovered when the length of the polygons' edges is taken to be vanishingly small. In this discrete approximation to a surface of given genus, different ways of dissecting the surface into polygonal tiles correspond to different possible configurations of the continuous surface.

To carry out the sum over surfaces needed to solve string theory and quantum gravity models, therefore, one needs a way to keep track of the possible ways of dissecting a surface. This is where the simple matrix models solved using the ideas discussed in 1978 come in. The dual to each Feynman diagram (or graph) of those models shows a way of dissecting a surface. (The dual to a graph or diagram is obtained by replacing each face of the graph by a point and joining those points by lines that cut the edges of the original graph just once.) If the model has a pth-order interaction—that is, if the interaction part of the model Lagrangian is a pth-order monomial in the matrix field—the duals to the Feynman diagrams are ways of dissecting the surface into p-sided polygons. Similarly, when interactions of pth and qth order are present, the dual graphs are dissections using simultaneously both pth- and qth-order polygons. (See the figure on page 17.) Furthermore, as 't Hooft had discussed already in 1974, the $1/N$ expansion of the models is an expansion in the genus of the surface: At a given order in $1/N$ one gets contributions from all Feynman diagrams that may be drawn without lifting the pen on a surface of genus equal to the order in $1/N$. (This explains why 't Hooft called the leading order of the $1/N$ expansion the planar approximation: Only diagrams that may be drawn on a sphere contribute in that order, and a sphere is topologically equivalent to a plane.) Therefore the term of order $(1/N^2)^r$ in the $1/N$ expansion of a matrix model having pth-order interaction generates all possible ways of dissecting a genus-r surface into pth-order polygons (see the figure above).

The three papers last fall that triggered the recent wave of exact solutions of models of string theory and quantum gravity showed, in essence, that the matrix models solved in the planar approximation in 1978 for all values of the coupling constants can be solved exactly for special values of their coupling constants. Those coupling constant values are the ones at which, in the language of dissections, the number of polygons of the given kind used in the dissection becomes very large, so that for a fixed area the surface ceases to be a fishnet and becomes continuous. In the matrix models, this is accomplished by taking two limits: The couplings of the models are taken to be infinitesimally close to the values at which each order in the $1/N$ expansion diverges, and simultaneously the limit $N \to \infty$ is taken. The necessity for such a double limit arises because the string coupling constant has a nontrivial dependence on the length scale over which it is probed.



First-order Feynman diagrams for the ground-state energy in a theory with quartic interaction. The diagram at top left, which is of leading order in $1/N$, may be drawn on a sphere or plane (lower left) without lifting the pen. But the same exercise for the diagram at right, which is of next higher order in $1/N$, may be done only on a torus. The double-line representation used here was introduced by 't Hooft to obtain the N dependence of the diagrams in matrix models. Each closed loop traversed as indicated contributes a factor N to the diagrams. (Adapted from D. Bessis, C. Itzykson, J.-B. Zuber, *Adv. Appl. Math.* **1**, 109, 1980.)

The genealogy of the ideas that have led to the exact string theory and quantum gravity solutions brings home the limitations of those solutions. The planar limit of the matrix models was solved only for a single space–time point, and the recent generalization of that solution also holds for a single space–time point. In the language of string theory, this means that the space–time in which the surface spanned by the string is embedded consists of just one point. The first solutions, however, have been extended in many ways. Douglas has obtained solutions for a space–time consisting of infinitely many points, and Brézin, Kazakov and Alexei Zamolodchikov (ENS Paris); Paul Ginsparg (Harvard) and J. Zinn-Justin (CEN Saclay); Gross and Nikola Miljković (Princeton); and Parisi have all solved models of strings embedded in one-dimensional space–time.

Excitement, nevertheless

Like many a development in mathematical physics, the new solutions also are obtained by solving differential equations. The differential equations arise from taking the continuum limit of the relevant sum over the discrete approximation to the surfaces. In equations for string models, the variable is the string coupling constant, so the equations describe how some string correlation function or vacuum energy or the density of excitations depends on that coupling constant. Fortunately, the differential equations encountered in the solutions all belong to a hierarchy of integrable equations generally referred to as the KdV hierarchy (after Korteweg and de Vries).

Solutions of differential equations usually depend on values of some parameters, which may represent initial or boundary conditions, for example. The new solutions also depend on parameters. Fixing the value of those parameters presented somewhat of a riddle in the beginning, but Brézin, Parisi and Enzo Marinari (University of Rome) have made some progress in this regard.

Witten has argued that the models solved exactly may be related to topological field theories. Another important question, Gross reminded us, is the relevance of the new solutions to the so-called critical string theories, such as the ten-dimensional superstring theories that in 1984 emerged as the strongest candidates for unifying all four fundamental interactions. —ANIL KHURANA

Reference

1. E. Brézin, V. Kazakov, Phys. Lett. B **236**, 144 (1990). M. Douglas, S.H. Shenker, Nucl. Phys. B. (in press). D. J. Gross, A. A. Migdal, Phys. Rev. Lett. **64**, 127 (1990). ■