DISORDERED ELECTRONIC SYSTEMS

Quantum mechanical coherence of electron wavefunctions in materials with imperfections has led to major revisions in the theory of electrical conductivity and to novel phenomena in submicron devices.

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The electrical conductivity of an ordinary metal such as gold is usually thought to be well understood. The electrons form a Fermi sea made up of plane waves modulated by the periodic crystal lattice. Because electrons obey Fermi statistics, only a narrow band of them, with an energy within $k_B T$ of the Fermi energy, contributes to the conductivity. At room temperature these electrons are scattered by lattice vibrations, resulting in a loss of momentum and a nonzero resistivity $\rho(T)$. At low temperatures electron–electron scattering is the dominant scattering mechanism. In the limiting case of zero temperature there is a residual resistivity ρ_0 caused by the scattering of the electrons at the Fermi energy by lattice imperfections such as impurities and vacancies. The static defects that disrupt the translational symmetry of the crystalline lattice are the source of the disorder considered in this article.

The resistivity is conventionally described by the equation

$$\rho(T) = \rho_0 + AT^n \qquad (1)$$

The exponent n is 2 for electron–electron scattering and ranges from 3 to 5 for electron–phonon scattering. The constant A is positive. The above picture is based on a free-electron model—that is, a model in which the states at the Fermi energy are assumed to be describable by the wavefunction for a single particle in an external potential. Indeed, it has generally been believed that Coulomb interactions between the electrons do not qualitatively modify the picture. The basis of this belief is Lev Landau's 1956 theory of Fermi liquids. That theory shows that Fermi statistics and phase-space restrictions give rise to a one-to-one correspondence between the low-energy excitations of an interacting Fermi system and those of the noninteracting system. The excitations of the interacting system can thereby be described by single-particle wavefunctions. Even though Landau's theory was derived only for translationally invariant systems, its applicability to disordered systems was largely unquestioned.

It is only within the past ten years that we have learned that almost everything in the above description about the resistivity of metals is wrong in the limit of low temperature. Interactions between electrons in disordered systems do lead to important corrections of the Fermi-liquid theory: The exponent n in the above equation is $\frac{1}{2}$ instead of 2, and the coefficient A can be of either sign, depending on details of the interaction.[1] Even more surprising is the discovery that in two-dimensional systems such as thin films, the residual resistivity ρ_0 in fact tends to infinity, albeit in a logarithmic way with decreasing temperature.[1,2] For ordinary metals these effects become observable only at extremely low temperatures.

As the amount of disorder is increased, however, these considerations dominate the physics. For example, if germanium is added to gold, the residual resistivity ρ_0 increases until it overwhelms the contribution of phonon scattering even at room temperature. For such a material the corrections discussed above, which are small for ordinary metals, become the dominant feature of the resistivity $\rho(T)$. When the germanium concentration reaches 82% (so that the compound is more correctly described as the alloy $Ge_x Au_{1-x}$ with $x = 0.82$), the zero-temperature conductivity vanishes. For even larger x, the resistivity becomes greater with decreasing temperature and the alloy becomes an insulator. This phenomenon, known as the metal–insulator transition, has been under intensive study since the 1950s.[2] Because the transition occurs at zero temperature as a function of a parameter describing the amount of disorder—in this case x—the problem hinges on understanding the quantum mechanical ground state of the disordered electronic system. In the past few years a scaling theory of the metal–insulator transition has been developed in analogy with the scaling

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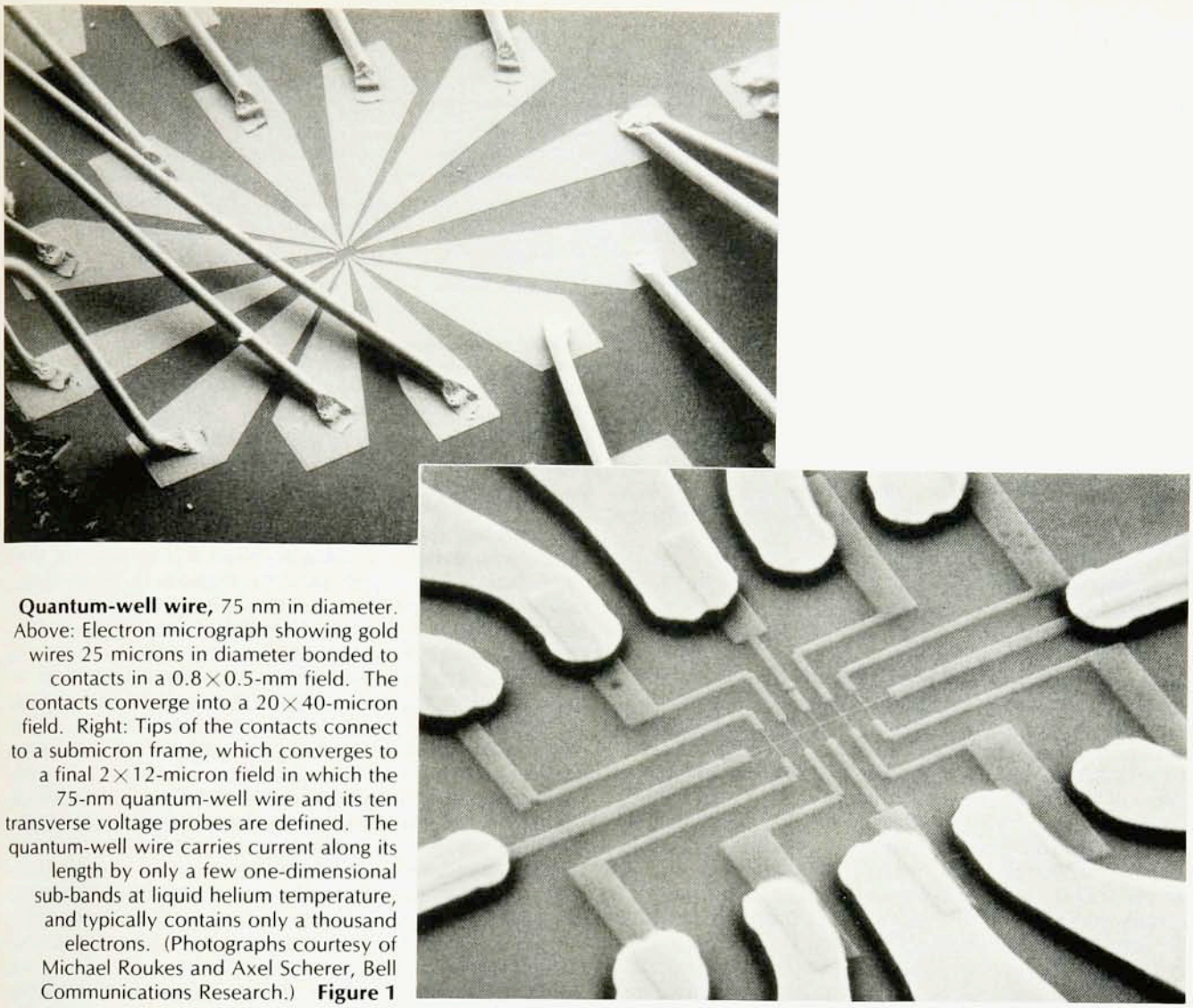


Quantum-well wire, 75 nm in diameter. Above: Electron micrograph showing gold wires 25 microns in diameter bonded to contacts in a 0.8×0.5-mm field. The contacts converge into a 20×40-micron field. Right: Tips of the contacts connect to a submicron frame, which converges to a final 2×12-micron field in which the 75-nm quantum-well wire and its ten transverse voltage probes are defined. The quantum-well wire carries current along its length by only a few one-dimensional sub-bands at liquid helium temperature, and typically contains only a thousand electrons. (Photographs courtesy of Michael Roukes and Axel Scherer, Bell Communications Research.) **Figure 1**

theory of second-order phase transitions at critical temperatures.

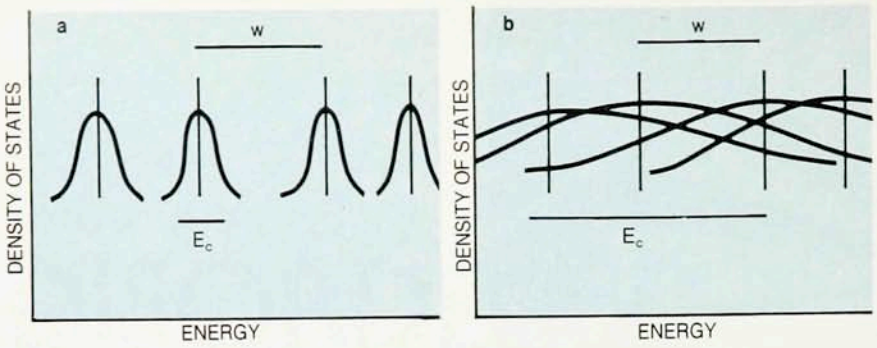
Localization of electrons

As a first step toward understanding the metal–insulator transition, we can take Landau's Fermi-liquid theory seriously and consider a model of noninteracting electrons scattered by the random potential due to the disorder. This is known as the Anderson localization problem. If the disorder is weak, then on a short length scale the wavefunction will look like a plane wave, but on a long length scale it will be scattered by the random potential. The distance over which the phase of the wavefunction deviates from that of the plane wave is called the mean free path. The multiply scattered wave is expected to have an amplitude everywhere in the sample, just as a plane wave does, and is referred to as an extended state.

Thirty years ago Philip W. Anderson pointed out that if the disorder is made progressively stronger, we should expect a qualitative change in the nature of the wavefunction.[3] In the limit of very strong disorder, we can envision the potential as a distribution of very deep potential wells, and the wavefunction will take the form of bound states that decay exponentially away from the binding sites. States whose amplitudes decay exponentially away from a center are called localized states. Physically one expects states to become localized when the mean free path becomes comparable to the wavelength. The central question in the Anderson localization problem is precisely how the localized states evolve from extended states as the disorder is increased.

The Anderson localization problem produces a metal–insulator transition because a system is a metal or an insulator depending on whether the states with energy at the Fermi level are extended or localized. However, this transition is not the full story for a disordered electronic system such as Ge_xAu_{1-x}. The interaction between the electrons turns out to be enhanced in the presence of disorder, and the corrections to the Fermi-liquid behavior in good metals that we discussed earlier rapidly take over the physics in the vicinity of the metal–insulator transition. We shall describe below some of the advances that have taken place in the theories of Anderson localization and interaction effects, and the way these two lines of investigation have deepened our understanding of the metal–insulator transition.

The findings about the onset of strong disorder have forced us to modify the traditional description of conductivity. Remarkably, our conventional views are being challenged from yet another direction. The rapidly advancing technology of submicron devices has led to the production of smaller and smaller structures. (One such structure is shown in figure 1.) These devices, often called mesoscopic systems, are large on the atomic scale, but



Schematic energy spectrum of an electron in a disordered solid. The eigenstates have a width E_c and an average spacing w. The width is larger when the sample is open to the outside world and the electron has the possibility of escaping. The dimensionless conductance g, given by E_c/w, is less than 1 in **a**, where the eigenstates are localized, and greater than 1 in **b**, where the eigenstates are extended. **Figure 2**

sufficiently small that the electron wavefunction is coherent over the entire sample. The condition for coherence is that the electron traverses the wire without undergoing any inelastic collisions with phonons or other electrons. For small wires such as those in figure 1 this condition is satisfied at liquid-helium temperatures. Because the electron can lose energy and equilibrate with the heat bath only via inelastic collisions, we have had to reexamine the conventional concept of energy dissipation in a resistor.

The standard framework for the description of conductivity has been the Kubo formula, which describes the linear response to an electromagnetic potential. For mesoscopic structures, where the resistivity is due entirely to scattering by static random potentials, and where inelastic collisions leading to dissipation occur mainly in the voltage and current contact pads outside the "sample," there is an alternative picture that is often more illuminating. In this formulation, by Rolf Landauer of IBM, the electrons are assumed to be in thermal equilibrium with various chemical potentials in the leads.[4] The sample, with its static random potential, is regarded as a scattering center for the electrons originating from the current leads, and the conductance, defined as the current divided by the voltage drop, is proportional to the transmission coefficient of the scattering problem. The proportionality constant, e^2/h, is an important combination of fundamental constants that we shall return to shortly. The Landauer formula was originally derived for a strictly one-dimensional geometry, but has been extended by Markus Büttiker of IBM to the multiprobe, multichannel case.[5] In its extended form the formula permits a satisfactory description of resistance even for extreme situations such as the one presented by the submicron wire shown in figure 1. The Landauer–Büttiker formula relates a nonequilibrium property, namely the conductance, to a scattering problem and thus has great advantages in terms of conceptual understanding and numerical computation. The Kubo formula, on the other hand, is more convenient for analytic calculations. The two approaches, which correspond to partitioning the electron–chemical potential into purely electromagnetic or purely chemical potentials, respectively, are equivalent and complement each other.

Studies of mesoscopic structures have revealed sample-specific changes in conductance, which are surprisingly sensitive to magnetic fields and even to the motion of a few impurity atoms in the sample. Phenomena such as these sensitivities are consequences of quantum mechanical coherence, and the combined efforts of theorists, experimenters and technologists in the past few years have led to rapid progress understanding them. Even though most of the theoretical results were discovered through complicated calculations, much of the physics underlying localization and interaction, as well as mesoscopic physics, is simple and intuitive. We will focus below on two important concepts: the idea of dimensional conductance as a key variable, and the picture of electron waves as random walking Feynman paths.

Dimensionless conductance

The problem that Anderson introduced and studied 30 years ago is deceptively simple: What is the nature of the eigenstates of a single particle moving in a random potential? The definition of a random potential is that at any point in space the potential might take any of a range of values, and only the probabilities for various values are specified. Anderson pointed out that there should be a qualitative change in the nature of the wavefunctions as the distribution of the potential is made broader.[3] (The width of the distribution increases with increasing disorder and is a measure of the disorder.) When the potential has a broad distribution, there is a significant probability that there are deep potential wells in which an electron might get trapped. Wavefunctions corresponding to these trapped states decay exponentially away from the binding sites. States whose amplitudes decay exponentially away from a center are called localized states, while states such as plane waves that have a nonzero amplitude everywhere in the sample are called extended states.

Many of the important physical ideas of the modern scaling theory of localization come from work in the early 1970s by David Thouless and his coworkers.[6] In the approach pioneered by Thouless, one considers hypercubes of linear size L in a d-dimensional space. Imagine that we have somehow diagonalized exactly the Hamiltonian for a single particle interacting with a random potential in this hypercube and that the average spacing w between single-particle energy levels is $(N_0L^d)^{-1}$, where N_0 is the number of energy levels per unit volume and per unit energy. Now imagine increasing—that is, scaling up—the sample size to $2L$ by putting together hypercubes of size L. The eigenstates obtained for each hypercube separately will no longer be exact eigenstates of the larger system. They will therefore decay on a time scale τ_D given by L^2/D, the time it takes a particle to diffuse across a hypercube and escape; D is the diffusion constant. We may use the uncertainty principle to associate a width E_c given by $\hbar/\tau_D$ with each eigenstate. If, as shown in figure 2a, the width associated with an "eigenstate" is much smaller than the separation between energy levels, the eigenstates in adjoining hypercubes will not mix appreciably. In other words, eigenstates of the combined system in this energy range will be confined mostly to one of the subsystems and will be well approximated by a superposition of the eigenstates of the subsystems. As we iterate this argument for hypercubes of increasing linear size, by building them from smaller hypercubes, we find that eigenstates of energies for which the condition $w \gg E_c$ is satisfied are increasingly confined to small regions of the large hypercube—that is, they are

localized. On the other hand, we can see that if $w \ll E_c$, as shown in figure 2b, the eigenstates corresponding to a hypercube become strongly admixed with those of other hypercubes, so that the eigenstates of the combined system will have large amplitudes in each of the subsystems. The wavefunction is therefore extended.

Through this kind of reasoning, Thouless argued that the nature of the wavefunctions in Anderson localization depends on the ratio E_c/w as the length scale is increased. It turns out that this ratio is related to the conductance G of the sample by

$$G = (e^2/h)(E_c/w)$$

Note that unlike the conductivity σ, the conductance, which is defined as the ratio of the current to the voltage drop across the sample, is a property of the finite sample and not an intrinsic property of the material. The prefactor e^2/h has the dimension of conductance—it is about $4\times10^{-5}\ \Omega^{-1}$—and is now even more famous as the unit by which the Hall conductance is quantized in the quantum Hall effect. We are thus led to define a dimensionless conductance g as $G/(e^2/h)$. The significance of g is that it is a scale-dependent parameter whose behavior at increasing length scales determines whether the eigenstates at the Fermi energy are localized or extended. This simple observation has a number of profound consequences, as we shall next discuss.

Thouless reasoned that because the conductance of a metallic wire is inversely proportional to its length, the dimensionless conductance of a sufficiently long wire should become small enough for the electron states to become localized.[7] Thus one reaches the remarkable conclusion that a sufficiently long metallic wire will not conduct electricity! This picture, however, is limited to zero temperature in that it is predicated upon the electron wavefunction's maintaining phase coherence across the length of the wire. Phase coherence is destroyed only by inelastic scattering, not by the elastic scattering caused by the static disorder, so the effect of finite temperature is described by introducing a phase relaxation time τ_ϕ, which is the time between successive inelastic processes. Because the electron motion in a disordered solid is diffusive, the average distance over which the electron diffuses during the time τ_ϕ and over which the phase coherence is maintained is given by the length

$$L_\phi = (D\tau_\phi)^{1/2}$$

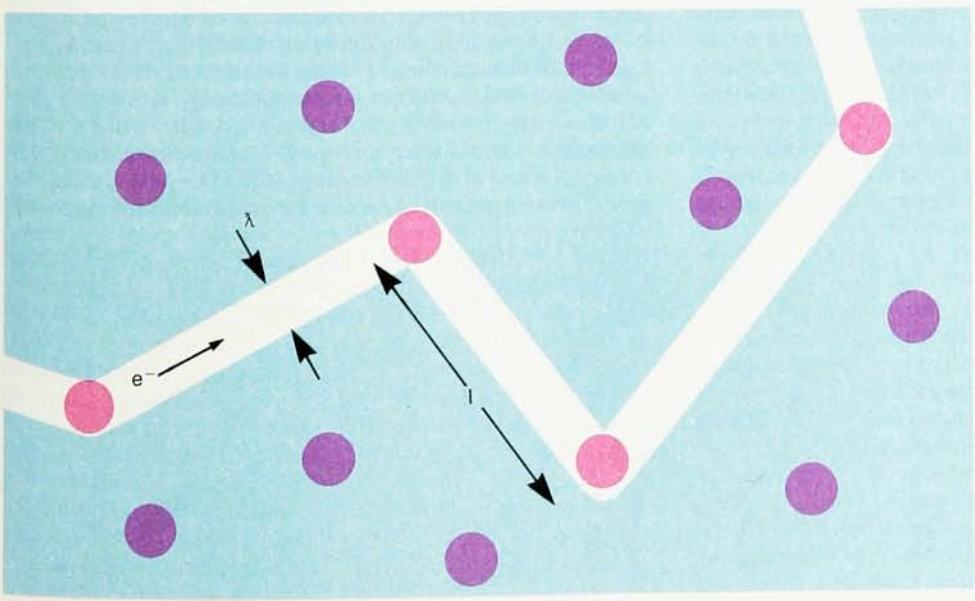
A necessary condition, then, for the localization of electron states in the wire is $L \lesssim L_\phi$. The inelastic scattering rate in general vanishes at zero temperature, so the length L_ϕ can become very long at low temperatures. Typically it can be as large as several microns below 1 K. Thouless estimated τ_ϕ for electron–phonon scattering and, based on his estimate, proposed that the resistivity of a wire would rise sharply below a certain temperature. This remarkable proposal led a number of groups to fabricate small wires and thin films and study their conductivity at low temperatures. These attempts resulted in the discovery by Gerald Dolan and Douglas Osheroff of a $\log T$ increase in the resistance of Pd–Au films, which was the beginning of a series of unexpected discoveries in the physics of micron-sized devices.[8]

A second important consequence of the concept of dimensionless conductance is that it led to the scaling theory of localization and the discovery of weak localization effects. The early steps were taken by Franz Wegner, who formulated a scaling theory in analogy with critical phenomena in which the dimensionless conductance plays the role of the scaling variable.[9] If the metal–insulator transition is like a critical point, it follows naturally that the conductivity should vanish continuously as the impurity concentration approaches the critical concentration n_c. This is because there must be a diverging length $\xi \sim |n - n_c|^{-\nu}$ near the critical point, and just from dimensional argument, we expect the conductivity σ to be $\sigma \approx (e^2/h)g/\xi$. Provided g scales to a fixed point g^* at criticality, we will have

$$\sigma \approx (n - n_c)^\mu \qquad (2)$$

where $\mu = \nu$. Equation 2 is at variance with the prevailing view at the time, that the conductivity should jump discontinuously at the metal–insulator transition, and the size of the jump was referred to as the minimum metallic conductivity σ_{min}. That prevailing view was based largely on analysis of experiments done at relatively high temperatures (above 1 K). A number of experiments done in the last ten years at temperatures as low as the millikelvin range support equation 2 instead.[2]

In a famous paper by Elihu Abrahams, Anderson, Donald Licciardello and T. V. Ramakrishnan,[10] a one-parameter scaling theory of localization was formulated



Typical Feynman path executing a random walk through a solid. In Feynman's formulation of quantum mechanics, the amplitude for the propagation of a particle between two points is given by the sum of the classical amplitudes over all the paths connecting the two points. Here the Feynman paths consist of randomly bent tubes of width $\lambda/2\pi$ whose straight sections connect pairs of strong impurity scatterers (circles). The mean free path ℓ is the typical step size. **Figure 3**

and explained in a lucid manner by postulating that the dimensionless conductance g is the only scaling variable. In support of this view, a diagrammatic perturbation theory was done that produces in two dimensions a logarithmically divergent correction to the conductivity, even in good metals, that is, when the disorder is relatively weak. This discovery led to a series of remarkable phenomena, which go under the name of "weak localization." We would like to give a qualitative explanation of the origin of these phenomena, which brings us to the second of the two basic concepts that we would like to introduce: the picture of the electron wave as random walking Feynman paths.

Random walk of Feynman paths

In Richard Feynman's formulation of quantum mechanics, the amplitude for the propagation of a particle between two points is given by the sum of classical amplitudes over all the paths connecting the two points. This differs from classical mechanics, in which one need consider the most probable path only. For a free particle, the dominant paths lie within a straight tube connecting the end points whose radius is on the order of the de Broglie wavelength λ. Let us imagine that the impurity atoms are distributed randomly in space, and let their concentration be n_{imp}. For simplicity, let us consider the unitarity limit, in which the scattering cross section is of order λ^2. In that case the Feynman paths consist of randomly bent tubes whose straight sections connect pairs of scatterers (see figure 3). The average length of a straight section is the mean free path l, which can be estimated by comparing the volume of the tube, $\lambda^2 l$, with the average volume per impurity, $n_{imp}{}^{-1}$. Thus l^{-1} is approximately equal to $n_{imp}{}^{-1}\lambda^2$, in agreement with the result obtained with Boltzmann transport theory. It is also convenient to introduce the elastic scattering time τ corresponding to the time it takes the electron to travel the distance l.

Now consider the propagation of an electron from point A to point B. Waves taking different paths, because of their different lengths, will arrive at B with random phases, so the probability of finding the particle at B is well approximated by the sum of probabilities of the various paths. In other words, the interference terms average to zero, and the addition of probabilities leads to the Boltzmann description, or classical diffusion. This is no longer true, however, when we look at closed loops, that is, when we consider the probability of returning to the starting point. Then, two paths of the same length (see figure 4) will interfere constructively. Their contribution in a quantum theory is therefore twice the value in classical diffusion. The enhanced probability of return decreases the diffusion constant and diminishes the conductivity. One can estimate the enhancement of backscattering by asking what fraction of the paths will return to the starting point within the time τ_ϕ, after which coherence is lost by inelastic processes. That estimate leads to the result that in three dimensions the relative decrease in the conductivity is given by

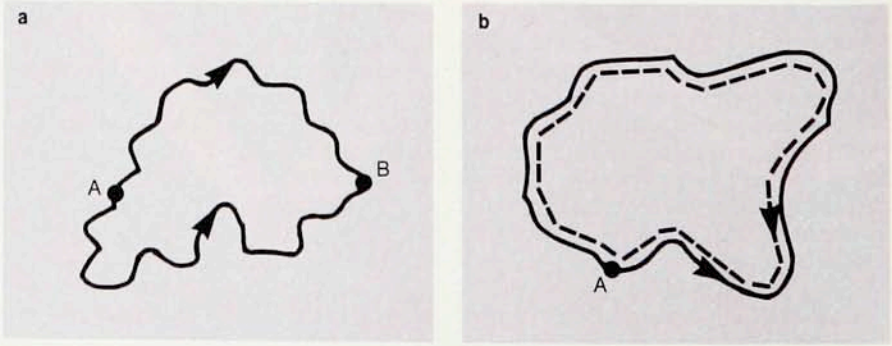
$$\frac{\delta\sigma}{\sigma} \approx \frac{-\lambda^2}{l^2}\left(1 - \sqrt{\frac{\tau}{\tau_\phi}}\right) \tag{3}$$

The correction is dominated by short trajectories and is finite in the limit $\tau_\phi \to \infty$, but it is small when l is large, that is, when the amount of disorder is small. For a thin film whose thickness a is much less than τ_ϕ, by contrast, trajectories of all lengths contribute equally, and the correction diverges in the limit $\tau_\phi \to \infty$ even when the disorder is weak. The correction for the conductivity per square of the film, defined as $\sigma_2 = a\sigma$, is

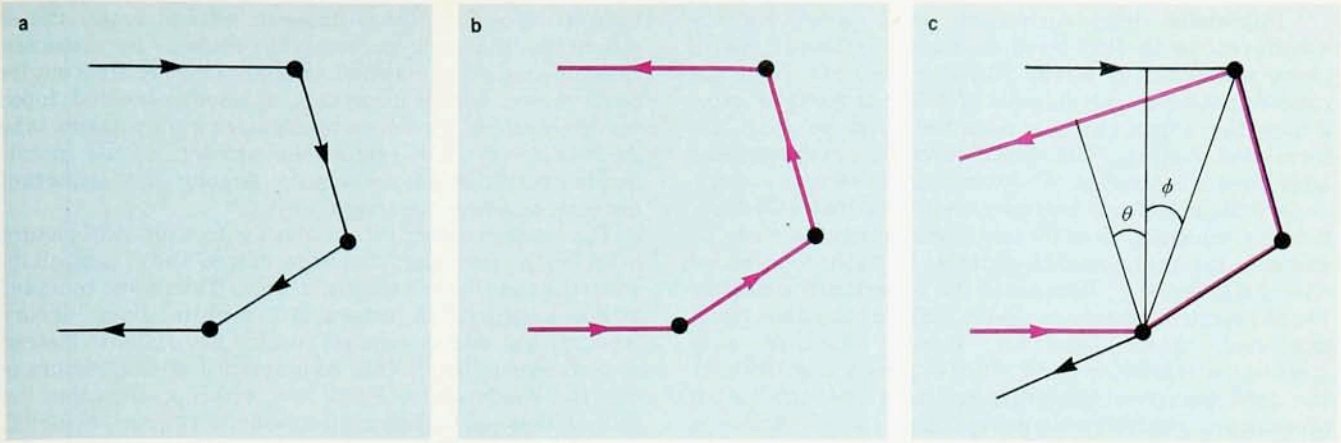
$$\delta\sigma_2 = (e^2/\pi h)\ln(L_\phi/a) \tag{4}$$

Here we have supplied the numerical coefficient from a diagrammatic calculation. This coefficient is a universal number, independent of the amount of disorder. For inelastic scattering, $\tau_\phi{}^{-1}$ typically is proportional to T^p, where the power p depends on which scattering—electron–phonon or electron–electron—is dominant in a given temperature range. Thus a ln T correction is predicted for the conductivity of thin films. The logarithmic divergence is another manifestation of the scaling-theory result that 2 is the marginal dimension in localization.

Thus we see that weak localization results from the interference between a closed Feynman path and its time reversed path, which we shall refer to as the conjugate path. This nomenclature is useful because it brings up the optical analog of the weak localization phenomenon. Consider a laser beam incident on a strongly scattering medium, such as a random collection of dielectric spheres. In figure 5a we illustrate the multiple scattering of the incident beam by three such scattering centers. To calculate the intensity of the backscattered light we have to add the amplitude from the conjugate wave shown in figure 5b. These two waves will be in phase because they have equal path lengths and the backscattering intensity is doubled. In contrast, away from the backscattering direction by an angle θ, as shown in figure 5c, the path difference between the wave and its conjugate is given by $d(\sin(\theta+\phi) - \sin\phi)$ where d is the distance between the first and last scattering centers. The interference becomes unimportant when this is comparable to the wavelength λ. Thus we expect a doubling of the backscattering intensity in a cone with angle $\theta \approx d/\lambda \approx l/\lambda$, since typically d is on the order of the mean free path l. It turns out that this phenomenon of coherent backscattering was discussed theoretically as early as 1969, but the quantitative observation was made only recently.[11] The results from one of the groups is shown in figure 6. The physical picture shown in figure 5 can be considered the reformulation of the returning Feynman path picture in momentum space, and has been explained and exploited



Propagation of an electron from one point to another in a solid. **a:** Two typical Feynman paths describing the propagation from A to B. **b:** Solid and dashed curves represent a Feynman path and its time-reversed path, which interfere constructively, enhancing the probability of return to the origin. This decreases the diffusion constant and the conductivity. **Figure 4**



Backscattering of light. **a:** Wave of light interacting with three scattering centers in such a way that the incoming and outgoing waves are parallel. **b:** Time-reversed, or conjugate, wave, which interferes with that shown in **a**. **c:** Scattering in a direction θ away from 180° backscattering. Small values of θ correspond to the region of enhancement. There is a phase shift between the path and its conjugate. **Figure 5**

experimentally by Bergmann in designing beautiful transport measurements in thin films.[12]

The most dramatic of the weak-localization effects in electronic systems occurs when a magnetic field is applied, in which case theory predicts a negative magnetoresistance. In standard transport theory, a magnetic field perpendicular to the current tends to cause the carriers to drift in a direction transverse to both the current and the field, thus increasing the resistivity. Weak-localization theory, by contrast, predicts a decrease in the resistivity when a magnetic field is turned on, and at low temperatures the effect is predicted to occur in fields as small as a few tens of gauss.[2] In the presence of a magnetic field perpendicular to a metallic film, each Feynman path picks up an additional phase factor $\int \mathbf{A}\cdot d\mathbf{l}$, where the line integral is along the trajectory. Hence in figure 4, the path and its conjugate will pick up opposite phases in a magnetic field. When the paths return to the starting point, their phases will be different by $2\oint \mathbf{A}\cdot d\mathbf{l}$—that is, by twice the flux enclosed by the loop the closed path forms—and the constructive interference leading to the weak-localization effect will be destroyed. Therefore the resistivity will decrease when the phase difference becomes of order π. The typical area of a loop is $L_\phi{}^2$, so that this condition can be described as $HL_\phi{}^2 \approx \phi_0$, where $\phi_0 = hc/e$ is the flux quantum. Because at low temperatures L_ϕ is much larger than the mean free path l, the field scale H becomes very small, on the order of tens of gauss as mentioned earlier. The negative magnetoresistance was first observed in the inversion layer of a silicon MOSFET and has since been seen in thin metallic films as well.[2,12]

An even more dramatic manifestation of the same physics is the prediction of oscillations in the magnetoresistivity of a sample shaped like a hollow cylinder.[13] Provided the walls of the cylinder are sufficiently thin and its perimeter is less than or on the order of L_ϕ, the phase difference between each Feynman loop that encloses the cylinder and its conjugate path will be the same and equal to $2\phi/\phi_0$, where ϕ is the total flux through the cylinder. Thus the conductivity along the cylinder is predicted to oscillate with increasing magnetic field, and the period of the oscillation is $\phi_0/2$, which is the flux quantum corresponding to a charge $2e$. This effect was first observed at the Institute for Physical Problems in Moscow by the father-and-son team of D. Yu. Sharvin and Yu. V. Sharvin.[14]

Interaction effects

The Coulomb interaction between electrons in metals is generally not weak: The potential and kinetic energies of electrons in metals are of the same order of magnitude. On the other hand, most properties of metals can be described quite well by treating the electrons as a noninteracting gas of fermions. This is so because according to Landau's Fermi-liquid theory, one can describe strongly interacting Fermi systems—that is, Fermi liquids—in terms of excitations called quasiparticles, which interact only weakly even in a strongly interacting electron system. Quasiparticles are well defined if their decay width γ is small compared with their energy ϵ measured with respect to the Fermi level. Because of the Pauli exclusion principle for fermions, the phase space for the decay of quasiparticles into particle–hole pairs is limited and the decay width γ is proportional to ϵ^2. As a result, low-energy quasiparticles are well defined. The quadratic dependence of the scattering rate on energy leads to a T^2 correction to the resistivity caused by electron–electron collisions. Fermi-liquid theory makes allowance for a strong renormalization of the density of states of the quasiparticle excitations but nevertheless leaves the density of states as a smooth function of energy, just as in the noninteracting fermions.

Fermi-liquid theory is valid only for translationally invariant systems, in which momentum is a good quantum number for both the energy eigenstates and the quasiparticle states. The simplification the Fermi-liquid theory offers is therefore lost in disordered systems, which by definition are not translationally invariant. Until recently, however, the common belief was that the Fermi-liquid picture ought to hold as long as the disorder is relatively weak. Al'tshuler and Arkadij G. Aronov have shown that this is not the case.[1] They found that in three dimensions the electron–electron scattering rate varies as $\epsilon^{3/2}$ and the correction to conductivity varies as $T^{1/2}$ at low temperatures. Furthermore, the density of single-particle states has a singularity of the form $|\epsilon|^{1/2}$ at the Fermi energy. In two dimensions the modifications are even more serious. For example, the corrections to the conductivity go as $\ln T$, with a coefficient again of order e^2/h, that is, of the same order as the weak-localization correction given in equation 4. As mentioned earlier, the $\ln T$ correction in thin films was the earliest sign of anomalous transport properties in disordered metals.[8]

Physically these corrections come about because wavefunctions in disordered media are composed not of plane waves but of waves multiply scattered from the random potential. Eigenstates of different energies experience the same random potential, and so they are correlated in space. The spatial correlation enhances the amplitude of scattering of two electrons of similar energy. In fact the amplitude becomes singular as the difference between the energies of the two electrons goes to zero. To estimate the enhancement, consider two electrons whose energies differ by ϵ. Because of the uncertainty principle the two electron states are almost indistinguishable over a time scale shorter than $\hbar/\epsilon$. During this time their interaction amplitudes add. In the presence of disorder the particles move diffusively and may encounter each other again within the time $\hbar/\epsilon$. Thus it is reasonable to consider the coupling constant to be enhanced by the fraction of the Feynman paths that bring a particle back to the origin within $\hbar/\epsilon$. The enhancement factor is given by considerations leading to equations 3 and 4 but with τ_ϕ replaced by $\hbar/\epsilon$. As mentioned earlier, the corrections are singular in $|\epsilon|$, but they vanish in the limit of zero disorder, where the mean free path for elastic scattering becomes infinite.

Some very puzzling data in the literature on metallic glasses are found to fit very well a $\sqrt{T}$-law, which we noted above arises also in a theory of interaction and localization effects.[2] That theory's prediction for the density of states—a square-root singularity at the Fermi energy—has also been confirmed experimentally. This agreement between theory and experiment on the density of states resolves the longstanding problem of the "zero-bias anomaly" in the tunneling spectra of disordered metals.

Metal–insulator transition

Localization theory predicts that the diffusion constant, and therefore the conductivity, vanishes at the metal–insulator transition but that other physical quantities, such as the single-particle density of states, should stay finite and analytic. Tunneling experiments, which measure the density of single-particle states, show, however, that the density of states at the Fermi energy, as well as the conductivity, vanishes at the metal–insulator transition[2,15] (see figure 7). Clearly something besides localization, such as the interaction between electrons, must be taken into account.

We do not yet have a complete theory of the metal–insulator transition, but it has been possible in the past few years to develop a theory that takes account of both localization and interaction effects.[16] The new theory has another two important scaling variables in addition to the dimensionless conductance of the scaling theory of localization. These variables describe the amplitudes for singlet and triplet scattering. As in the theory of more usual types of phase transitions, the behavior at the metal–insulator transition can also be put into several universality classes, each associated with a different set of critical exponents. The behavior of the scaling variables—that is, whether the variables scale to zero, a constant or infinity—depends on the universality classes, and the physical realization of each universality class in turn depends on the strength of spin-flip scattering, spin–orbit scattering and magnetic fields.

Experiments on alloy films such as $Au_{1-x}Ge_x$ and $Nb_{1-x}Si_x$ indicate that the conductivity exponent μ defined in equation 2 has a value close to unity (see figure 7). By contrast, the exponent is close to 0.5 in uncompensated silicon doped with phosphorus. Very recent experiments have shown that the value of the exponent varies between 0.5 and 1 as the compensation is introduced. It is natural to explain these different critical exponents as describing different universality classes—for instance, spin-flip scattering may be important in alloy films but for some reason not so important in uncompensated doped semiconductors. However, much more work remains to be done before we can regard the problem of the metal–insulator transition as solved. Among the important unresolved issues are the following:

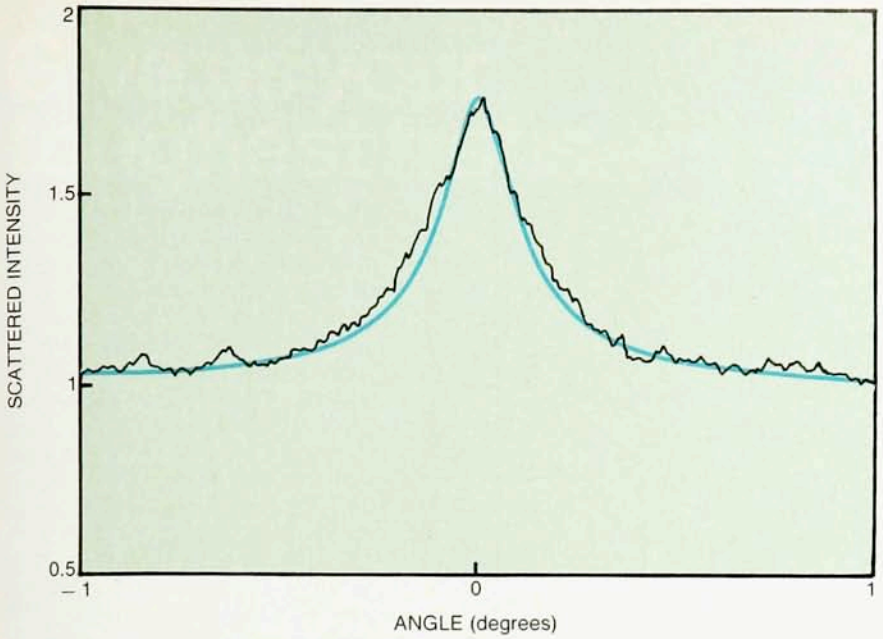
▷ The present theory is consistent with a physical picture in which the transport of charge, of spin and of heat all involve the same quasiparticle carrier. Thus some remnant of the quasiparticle picture of the Fermi liquid theory remains, but with quasiparticles that are diffusive instead of plane-wave-like.[17] One consequence of this picture is that the Wiedermann–Franz law, which predicts that the ratio of thermal to electrical conductivity is $(\pi^2/3)(k/e)^2T$, should hold on the metallic side all the way up to the metal–insulator transition. This prediction and, more generally, the existence of quasiparticles remain to be tested.

▷ The existence of local magnetic moments on the metallic side of the metal–insulator transition is a longstanding problem that the current theory does not really address. Formation of the local moments may involve states very far in energy from the Fermi energy. Such states are outside the scope of the scaling theory, which considers states near the Fermi level only. Nuclear magnetic resonance experiments and very recent simultaneous measurements of magnetic susceptibility and specific heat down to very low temperatures in phosphorus-doped silicon provide strong evidence for local moments on the metallic side of the transition.[18] These local moments might control the amount of spin-flip scattering, which in turn determines the universality class to which a system belongs.

Mesoscopic systems

Recent developments in lithographic technique have made it possible to fabricate systems in the submicron range. Because at millikelvin temperatures L_ϕ is on the order of a micron or more, it is possible at those temperatures to study samples whose dimensions are smaller than or comparable to L_ϕ. The quantum mechanical coherence of the electron waves dominates the physics under these circumstances. This opens up a new field of physics, broadly referred to as mesoscopic systems, that deals with structures large on the atomic scale but small enough that quantum mechanical coherence is important and macroscopic averaging of the type familiar in statistical physics does not describe the behavior of a sample. These systems are characterized by sample-specific and reproducible fluctuations. The property that has been studied in greatest detail is fluctuations in the conductivity of small structures as a function of magnetic field, chemical potential and other parameters. Richard Webb and Sean Washburn, in their article on page 46, describe experimental work and recent developments in mesoscopic systems, and Yoseph Imry has written an excellent review of the theoretical concepts.[19]

The origin of the fluctuations can be understood by starting from Landauer's formula, which relates the conductance to the transmission probability.[4] The probability amplitude that an electron is transmitted from one side of the sample to the other is proportional to the sum of amplitudes of Feynman paths that walk randomly across the sample. In the presence of a magnetic field, each Feynman path will acquire a phase $\int \mathbf{A}\cdot d\mathbf{l}$, and the interference between two typical paths will be completely altered if the flux enclosed by the two paths is of the order ϕ_0. The area enclosed by the two paths is typically the



Intensity of backscattered laser light incident on a concentrated aqueous suspension of latex microspheres. Note the enhancement by almost a factor of 2 in the 180° backscattering direction. Theory is shown by the colored curve. (Adapted from E. Akkermans, P. E. Wolf, R. Maynard, Phys. Rev. Lett. **56**, 1471, 1986.) **Figure 6**

same as the area of the sample. Therefore this picture predicts that the conductivity will fluctuate when the magnetic field is changed by ΔH, where the product of ΔH and the area is of the order of the flux quantum ϕ_0. This prediction agrees with experimental as well as numerical simulations.

This argument, however, does not give any information about the size of the fluctuation, and one might expect that the fluctuation will average to zero because there are many Feynman paths, especially as the sample size increases. It turns out that there is a remarkable universality in the magnitude of the fluctuation, namely that the conductance of the sample should fluctuate by order e^2/h, independently of sample size, dimensionality and the amount of disorder, provided the disorder is weak enough to be far from localization, and the temperature sufficiently low.[20] Many experiments have confirmed this idea of universal conductance fluctuation and its extension to finite temperatures.

To put this result in context, we note that for finite samples it is natural to expect sample-to-sample fluctuations in any physical quantity, including the conductance, just because each sample contains a different distribution of impurities. However, standard statistical fluctuations obey the law of large numbers, so that in α-dimensional space the relative fluctuation is expected to obey

$$\langle \delta G^2 \rangle / \langle G \rangle^2 \approx (L_c / L)^d \tag{5}$$

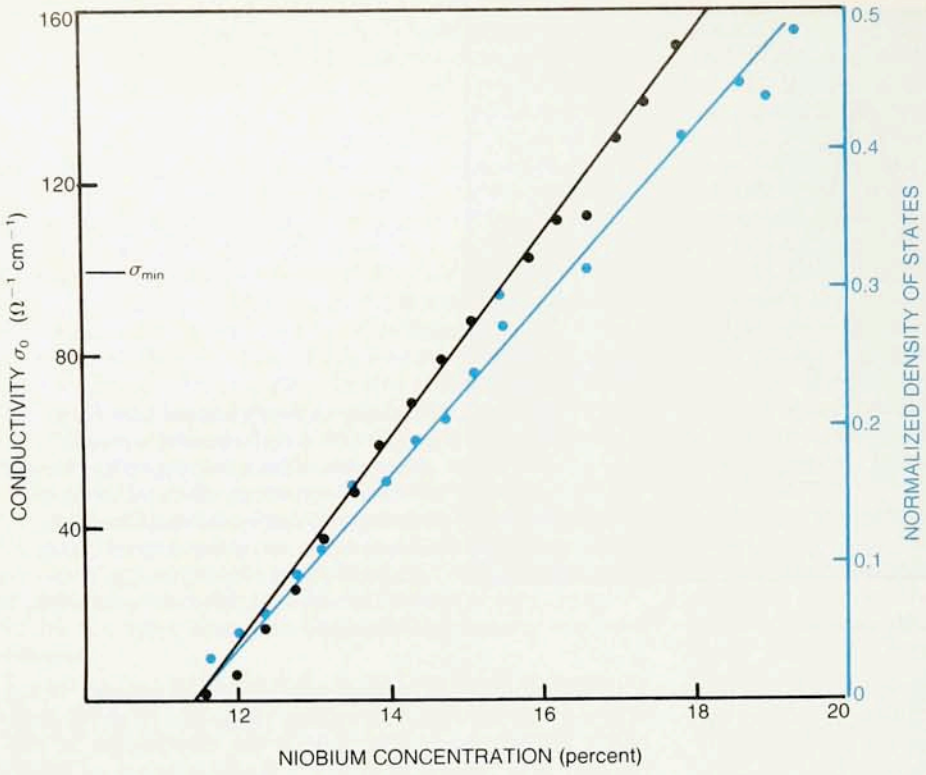
Here L_c is a coherence length that one might naively associate with a microscopic length such as the mean distance between impurities. It turns out that because of quantum coherence effects, L_c is the mesoscopic length scale given by L_ϕ, so that equation 5 is obeyed only for $L \gg L_\phi$. For $L \lesssim L_\phi$, and using the fact that from Ohm's law $\langle G \rangle \sim L^{d-2}$, we see that the universal conductance fluctuation $\langle \delta G^2 \rangle$, which is approximately equal to e^2/h, implies that $\langle \delta G^2 \rangle / \langle G^2 \rangle \sim L^{4-2d}$, so that for $d < 4$ the fluctuation is much larger than that given by equation 5. In particular, in two dimensions, even the relative fluctuation is independent of size (until localization sets in), indicating that the system does not self-average.

If we look at the conductance fluctuation from a quantum mechanical point of view, we reach the opposite conclusion, namely that the fluctuation is smaller than naive expectations would dictate. According to equation 1, the dimensionless conductance is just the number $N(E_c)$ of energy levels within an energy range E_c. If the energy levels are randomly distributed, the fluctuation in the number $N(E_c)$ would be $N(E_c)^{1/2}$, much larger than unity, the correct answer. This apparent discrepancy can also be seen from figure 2b, where superposition of a number of broadened but randomly located eigenvalues leads to large fluctuations in the density of states. The resolution of this discrepancy is that the distribution of eigenvalues of a random Hamiltonian is not random, because eigenvalues tend to repel each other. Freeman Dyson (Institute for Advanced Study) studied this phenomenon of spectral rigidity and showed that the rms fluctuation in $N(E)$ goes as $\ln N(E)$, much less than $N(E)^{1/2}$ according to random statistics. Dyson obtained his result by counting the number of states in a precise energy window of width E. In our case the finite width of the levels makes it reasonable to smear the edge of the window by a width of order E_c, and it can be shown that in that case Dyson's result becomes unity, in agreement with the universal conductance fluctuation result.[21]

The remarkable sensitivity of the conductance of a sample to an external magnetic field raises the following question: How sensitive is the conductance to a small variation in the random potential? This question was answered by Al'tshuler and Boris Z. Spivak[22] and by Shechao Feng, Lee and A. Douglas Stone,[23] who concluded that if a fraction $\delta N/N_{imp}$ of the total number of impurities N_{imp} in a sample of size L are moved, the conductance will change by

$$\delta G \sim \begin{cases} e^2/h, & \text{if } \delta N/N_{imp} > l^2/L^2 \\ (e^2/h)(L/l)(\delta N/N_{imp})^{1/2}, & \text{if } \delta N/N_{imp} < l^2/L^2 \end{cases} \tag{6}$$

In particular, in two-dimensional films the motion of even a single impurity atom leads to a δG that is a fraction of e^2/h, no matter what the sample size L. This surprising result can also be understood from the picture of random walking Feynman paths. From figure 3 we can see that each Feynman path across the sample visits $(L/l)^2$ impurity sites, so that the fraction of sites visited is $(L/l)^2/N_{imp}$. It follows that a given site is visited by the same fraction $(L/l)^2/N_{imp}$ of all the Feynman paths. The motion of an impurity changes the phases of all the Feynman paths it visits. To change phases of all the Feynman paths traversing the sample, we have to move



Metal–insulator transition. The compound Nb_xSi_{1-x} becomes a metal when the concentration of niobium atoms exceeds about 11.5%. Black circles represent conductivity data and refer to the left-hand axis. Colored circles represent the normalized density of states $N(0)/N(100\ mV)$ at the Fermi level and refer to the right-hand axis. Note that the conductivity vanishes gradually with x, not abruptly, and that it takes on values much below σ_{min}, contradicting older theory. Note also that the density of states vanishes together with the conductivity at the metal–insulator transition. (Adapted from G. Hertel, D. J. Bishop, E. G. Spencer, J. M. Rowell, R. C. Dynes, Phys. Rev. Lett. **50**, 743, 1983.) **Figure 7**

$N_{imp}(l/L)^2$ impurities, in agreement with equation 6.

The sensitivity of the conductance to the motion of a single impurity opens up the possibility of using conductance to probe slow dynamic processes, such as the diffusion of impurities or the dynamics of spin glasses or two-level systems. Already, fluctuations on a time scale of seconds to minutes have been observed in bismuth films below 1 K and have been interpreted as the result of some kind of atomic rearrangement in the film. Slow configuration change is known to lead to $1/f$ noise in metals; the suggestion had been made that in metallic glasses the two-level system becomes the dominant source of configuration change at low temperatures, so the increase in the $1/f$ noise with decreasing temperature was predicted.[23] Recent experiments on copper–carbon and bismuth films[24] have apparently confirmed this prediction and have found that the $1/f$ noise below 50 K is due to this kind of fluctuation. Thus the quantum interference phenomenon is not necessarily restricted to submicron samples in the millikelvin temperature range.

What we have learned about the wave nature of electrons in disordered media should enhance our understanding of the propagation of classical waves such as electromagnetic or sound waves in these media. The coming years should bring further new developments as well as a deeper understanding of longstanding problems such as the metal–insulator transition.

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