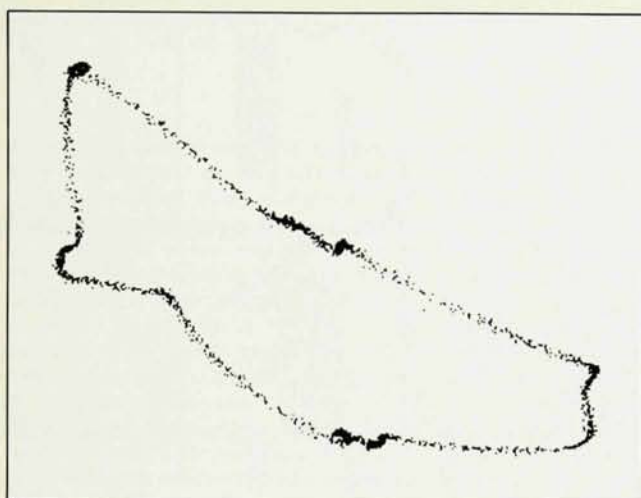


New global fractal formalism describes paths to turbulence

One motivation for the intensive study of nonlinear physical systems is the hope that, despite their complex structures, they might possess universal features shared by entire classes of similar nonlinear processes. This hope was strikingly realized several years ago when Mitchell Feigenbaum (then at Los Alamos, now at Cornell) discovered that a few universal ratios— independent of any dynamical details—characterized all systems whose periods doubled repeatedly as they approached turbulence (PHYSICS TODAY, March 1981, page 17). At the point of infinite period doubling, the orbits of Feigenbaum's system showed a complex behavior in which one could discern a scale-invariant, or fractal, structure. Recently a team of theorists has developed a method for describing a new global structure of these fractal objects. The predictions generated by this formalism agree quite well with experimental measurements performed at the University of Chicago on a fluid as it approached turbulence by two distinct paths.

The theoretical approach was developed¹ by Thomas C. Halsey, Mogens H. Jensen and Leo P. Kadanoff (University of Chicago), Itamar Procaccia (Weizmann Institute) and Boris I. Shraiman (Bell Labs). It is perhaps easiest to understand their work as it applies to the particular dynamical system on which it was tested² experimentally. The system was a forced Rayleigh-Bénard experiment, and the measurements were performed at Chicago by Albert Libchaber, Joel Stavans, James Glazier and François Heslot. Their apparatus consists of a container of mercury heated from the bottom and cooled from the top. When the temperature gradient becomes sufficiently large, the fluid develops two convective rolls, flowing up in the middle and down along the walls. With a stronger gradient, the convective rolls begin to oscillate such that an imaginary line between the rolls and parallel to their axes swings back and forth horizontally with small amplitude.

In addition to the natural frequency associated with this convective oscillation, the system is driven at a second



Critical orbit is the trajectory of a dynamical system in phase space as the system becomes turbulent. The orbit shown is the intersection of a distorted torus with a plane. It was determined² from temperature measurements of a mercury cell undergoing convective and electromagnetic oscillations.

frequency by an electromagnetic signal: A vertical ac current is applied to the center of the conducting mercury fluid while a dc magnetic field is imposed horizontally, parallel to the axes of the convective rolls. When the two frequencies are incommensurate, the interaction between them causes the system trajectory to trace out a torus in phase space. As the amplitude of the ac current increases and drives the system toward turbulence, the torus becomes greatly distorted. The intersection of this distorted torus with a plane, as determined by experimental measurements on the system near the onset of chaos, is shown in the figure on this page. This so-called critical orbit is derived from readings of the temperature of the system at discrete times separated by the period of the forcing.

One glance at the critical orbit shows that some regions of phase space are visited more frequently than others. The new formalism is aimed at describing these density variations, or the bunching of points in this critical orbit (or in similar structures generated by other approaches to chaos). It starts by expressing the probability p_i that other

points might fall within some distance l of a given point in terms of a scaling index $\alpha_i(l)$

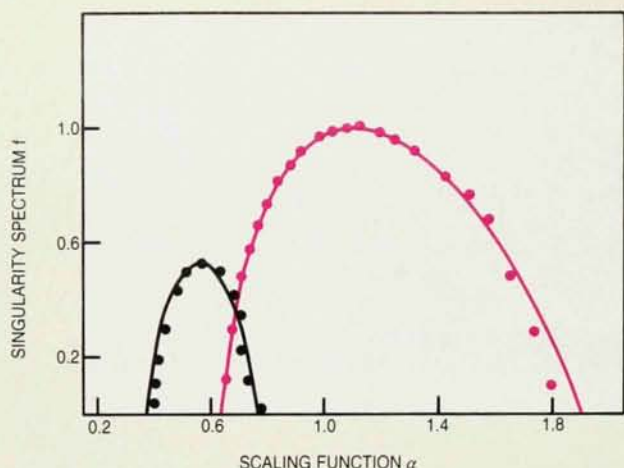
$$p_i = l^{\alpha_i(l)}$$

The scaling index α can assume a range of values. The distribution of values is called the spectrum of singularities and is denoted by the function $f(\alpha)$. This function roughly expresses how many times each kind of scaling occurs; one may therefore think of f as a kind of entropy. Thus the critical orbit is described as if it were composed of interwoven sets of singularities, each with a different value of α .

To describe a particular physical system with this formalism requires a model of that system. Several experiments on the forced Rayleigh-Bénard system have indicated that its critical orbit can be modeled well by a nonlinear mapping of the circle onto itself. That circle map has the form

$$\theta_{n+1} = \theta_n + \omega - (a/2\pi) \sin(2\pi\theta_n)$$

Using this map, the theorists predicted that the curve of f versus α would look like the red curve in the figure on page 18. Experimental data were similarly processed to yield an f vs. α curve,



Spectra of singularities, f , indicate roughly how often a particular value of the scaling index α occurs. The red spectral curve describes the critical orbit of the figure on page 17 for a forced Rayleigh-Bénard system approaching turbulence along a quasiperiodic route. The black curve describes the system approaching turbulence by period doubling.

representative points of which are depicted in red in the figure. The theoretical and experimental curves agree quite well. The spectrum shown in black shows similar agreement of the theory with experiment when it was applied to a forced Rayleigh-Bénard system that was approaching turbulence along a period-doubling route.

Starting in part with the pioneering studies of Benoit Mandelbrot (IBM) a decade ago, work on fractal systems has flourished in recent years. The basic idea of representing the global structure of fractals in terms of the formalism described here originated³ with the work of Uriel Frisch (Nice Observatory) and Giorgio Parisi (University of Rome) on well-developed turbulence. Later, Roberto Benzi (Centro Scientifico IBM, Rome), Giovanni Paladin (University of Rome), Parisi and A. Vulpiani (University of Rome) took the same ideas and used⁴ them to characterize the magnitude of the velocity at a point (α) and the number of points with that velocity (f). Halsey, Paul Meakin (DuPont) and Procaccia later explored⁵ the concept as it applies to aggregation phenomena, but they did not write f as a continuous function in that work.

The quasiperiodic route to chaos. The forced Rayleigh-Bénard system is one of a class of systems with two driving frequencies. If the two frequencies are commensurable, the system locks onto the initial frequency ratio over some finite range of the control parameter, in this case the amplitude of the ac current. A system driven in this manner may approach chaos along the period-doubling route, in which the period successively doubles as the control parameter increases. The black curve in the figure above corresponds to this case.

However, if the two frequencies (whose ratio is called the winding number) are incommensurable, such mode locking does not occur, by definition. The power spectrum of the system response in this case can be ex-

pressed as the sum of linear combinations of the driving frequencies, with the number of such combinations increasing as the system approaches turbulence. In this mode the system is said to take the quasiperiodic approach to chaos. This approach is depicted in the red curve of the figure above.

Scott Shenker (Xerox Palo Alto Research Center) first analyzed quasiperiodic systems phenomenologically in 1982. Soon thereafter Feigenbaum, Kadanoff and Shenker studied these quasiperiodic systems in more detail, as did a group consisting of Stellan Ostlund (University of Pennsylvania), David Rand (Warwick, England) and James Sethna and Eric Siggia (both of Cornell). They all modeled the system using the nonlinear mapping of the circle onto itself. These studies found that the transition to chaos occurs in a universal way when the winding number equals $(\sqrt{5} - 1)/2$, the classic irrational number called the "golden mean." This case is the easiest one to analyze theoretically and study experimentally.

Subsequent experiments on Rayleigh-Bénard cells with winding numbers fixed at the golden mean verified the universal properties predicted by these analyses. In a Rayleigh-Bénard cell with water as the fluid, Aaron Fein (now at IBM, Yorktown Heights) and Mike Heutmaker and Jerry Gollub (both at Haverford College) observed⁶ some evidence for spectral self-similarity; that is, they found that the scaled power spectrum had the regular pattern of peaks predicted by the theory. Stavans, Heslot and Libchaber performed⁷ a similar experiment, but they managed to have better control over the winding number by selecting mercury as the fluid and exciting it electromagnetically rather than thermally. They also found the regular spectral pattern, and they were able to measure a universal parameter as well. These measurements indicated that the experimentally observed

critical orbit could be represented well by the corresponding orbit in the circle map.

The current work on a universal formalism contrasts in some ways with Feigenbaum's discovery of universal properties in the period-doubling route to chaos. Kadanoff points out that much of the theoretical work to date has concentrated on detailed analysis of special points in phase space, whereas his formalism allows a global characterization of phase-space orbits. Feigenbaum suggests that the method introduced by Kadanoff and his colleagues, by seeking only coarse rather than detailed information about global properties, has a very good chance of being verified experimentally. He compares the f vs. α curve to a thermodynamic function in which one has summed over the microscopic information, which is also available, to obtain a quantity that can be measured easily. He also comments that, because the new approach is expressed in terms of a statistical-mechanics formalism, it forges some connections between chaotic physics and more standard physics. He is working on still more thermodynamic considerations.

Ostlund characterizes the method as providing a phenomenological description of any object, especially useful for a fractal with many scaling exponents. He feels it to be complementary to studies that calculate the specific exponents from a detailed renormalization theory. Indeed, the first papers to discuss the f vs. α formalism did not use renormalization theory, but rather assumed a given mapping and generated simulated data. Since then, Kadanoff, Jensen and David Bensimon (Chicago) have applied⁸ renormalization techniques to derive the same results from first principles.

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