Nuclear dynamical supersymmetry

Until recently, supersymmetry existed only as a theoretical concept; recent studies of atomic nuclei hint that it may be a physical reality.

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One of the goals of physics is to establish simple sets of laws that provide a unified picture of apparently different physical phenomena. In the realm of elementary particle physics, attempts have been made over the last decade to unify all known forces in nature by a common highest symmetry known as supersymmetry. In recent years these ideas have become quite widespread in various branches of physics, ranging from investigations of the early universe to polymer physics. This burgeoning and diversified interest was recently exemplified at the Workshop on Supersymmetry in Physics[1] at Los Alamos, New Mexico, in December 1983, which brought together physicists from many different disciplines.

Despite the appeal of supersymmetry schemes as unifying theoretical devices in physics, there has been little or no empirical evidence suggesting their validity. Recently Bruno Zumino, one of the first to develop the concept of supersymmetry, stated[2] that "supersymmetry is a general framework rather than a specific theory. What we need is an equally appealing specific model whose consequences could be tested experimentally." The recent postulation by Francesco Iachello,[3] that the concept of supersymmetry may also be applicable to atomic nuclei, alters this situation because, as we shall show, this is a realm wherein such concepts can indeed be extensively tested empirically.

The specific supersymmetry schemes relevant to nuclear physics differ somewhat from those used in other branches of physics, but the fact that it is possible to test their predictions in detail in nuclear-structure physics lends wider interest to this field. In this article we discuss supersymmetry and its empirical testing in certain heavy nuclei. The various experimental techniques used in this process represent a fine example of the richness of nuclear spectroscopy, for they involve completely different yet complementary approaches: probes that are selective for specific features of the nuclear structure and those that are not selective. In the last three decades nuclear physics has made enormous strides both in the accumulation of vast amounts of data on the properties and excitation modes of atomic nuclei and in the development of models to interpret these data. A longstanding central issue, both from the theoretical and experimental points of view, has been the study of "collective" states: modes of nuclear excitation in which many protons and neutrons participate simultaneously and coherently. They contrast with the so-called "single-particle" excitations in which only one nucleon (proton or neutron) alters its orbit in the nucleus. Although highly successful models that describe collective properties have existed since the 1950s, the appearance in the last decade of a new model—the interacting-boson model proposed[4] by Akito Arima and Iachello in 1974—has revitalized the field. The new model provides a unified description of seemingly different manifestations of collective motion in nuclei and a deeper understanding of the underlying dynamical symmetries (see the box on page 28 for a discussion of this concept in a familiar context). With use of the interacting-boson model, it is, in fact, now possible to describe succinctly the symmetry structure of broad ranges of nuclei. This is illustrated by the color-coded symmetry chart of the nuclei in

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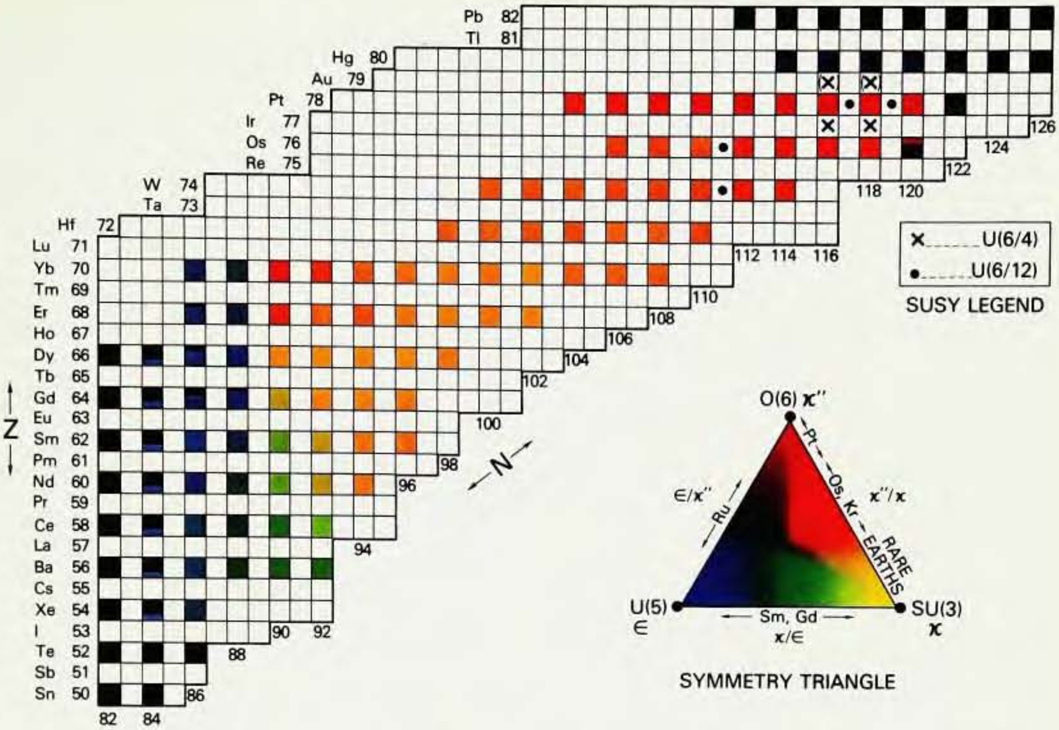


Symmetry structure of nuclei from mass numbers 140 to 208. In the lower right is shown the symmetry triangle for the interacting-boson model: Each vertex represents one of the three symmetries of the model, and the legs denote transition regions. Specific nuclei can be placed at the appropriate locations in the triangle. The colors shown in the triangle are used in the accompanying chart of nuclides to indicate the approximate symmetry characteristic of the well-studied nuclei in the rare earth and Os–Pt regions. The crosses and dots indicate nuclei that are thought to reflect one or another of the supersymmetries discussed in the article. (Thanks to David D. Warner for help in designing this figure) Figure 1

figure 1 that will be discussed in more detail below. Moreover, the interacting-boson model leads to new and surprising predictions that have inspired new measurements and that have, in turn, been confirmed by them. One of the most interesting aspects of this research is the proposition that the dynamical symmetries may also be extended to the wider concept of supersymmetry schemes and, furthermore, that the existence of such symmetries may be verified by experiments. In this article we describe this new model and the status of current supersymmetry ideas in nuclear physics. First, however, we present a brief discussion of nuclear models and of the interacting-boson model.

Nuclear models

A principal goal of nuclear physics is the construction of models that explain the observed structure and excitation modes of the atomic nucleus in terms of the behavior of its constituent protons and neutrons.

Traditional models. For the past 35 years the generally accepted standard has been the shell model proposed[5] in 1948 by Maria Mayer, Otto Haxel, Hans Suess and Hans Jensen, which in its simplest form envisages the nucleus as an assemblage of nucleons (neutrons or protons, considered independently) that orbit the nuclear center of mass in a mean central potential in a fashion analogous to the planetary model of electrons orbiting an atom in the Coulomb potential. Figure 2 shows some of the energy levels for these orbits for the 50th through the 126th nucleon. These orbits are labeled by several quantum numbers, one of which is the total angular momentum j of the orbit. To determine the properties of a nucleus—such as ground-state mass, excited-state energies and spin-parity quantum numbers (J^π)—is then extremely simple. The nucleons are placed in successive shell-model orbits, starting with the lowest one. The Pauli principle, which applies to all fermions (nucleons) only allows $2j+1$ nucleons in an orbit of total angular momentum j. In addition, the nucleon–nucleon interaction is such that pairs of protons or of neutrons tend to occupy orbits with equal but opposite angular momenta, resulting in a total angular momentum of zero. Thus the angular momentum, J, of a nuclear ground state with many nucleons is equal to 0 if the number of both protons and neutrons is even (even–even nucleus), and J equals the angular momentum j of the last unfilled orbit if either the proton or neutron number is odd (odd–even nucleus). Excited states in an odd nucleus can be formed by changing the orbit of this last nucleon. The extra energy required can simply be read off from figure 2. Note that the orbits tend to group into clusters or shells: just as in the atomic case. When the number of nucleons is such that one of these shells is filled, a particularly stable nucleus is formed, as is also true in the case of closed shells in atoms. In nuclei that have just a few nucleons in the unfilled shell (few "valence" nucleons), the simple picture just described is indeed verified. However, in nuclei with many valence nucleons, one frequently observes a different type of structure characterized by collective states that involve many nucleons in combined motions such as rotations or vibrations. Such states are widespread throughout the periodic table and they appear in both even–even and odd–even species. Thus they are extremely important to understand. In fact, much of the history of the study of medium- and heavy-mass nuclei in the last 20 years or so has centered on attempts to elucidate the structure and systematic properties of their collective states.

One method utilizes the shell-model approach extended to include the orbits, not just of the last unpaired nucleon, but of all those valence nucleons outside the "core," which is formed by the closed-shell nucleus. This approach is illustrated on the right in figure 3. In carrying it out, all the possible ways of constructing a given angular momentum from such multi-particle configurations must be considered. Unfortunately this can often lead to insurmountable problems, because the number of possible configurations rapidly becomes astronomical as the number of valence nucleons increases. To quote a famous example given[6] by Igal Talmi: in Sm^{154}, which has 22 valence protons and neutrons, the number of states with spin-parity assignment 2^+ alone amounts to 346 132 052 934 889.

An alternative approach is the formulation of models based on a geometric concept (analogous to the well-known liquid-drop model), proposed[7] in the early 1950s by Aage Bohr and Ben Mottelson. In these models the nucleus is assigned a shape and is allowed to undergo rotations or vibrations or both. This approach is appealing because it offers a simple visual picture and is also, in many cases, extremely simple to deal with. On the other hand, while geometrical schemes can interpret the excitations of a given nucleus once its shape is given, it is difficult in such schemes to predict *a priori* the behavior of any nucleus or, in particular, of a sequence of nuclei of neighboring masses. For this reason, other approaches have frequently consisted of attempts to truncate the full shell model by considering only a subset of available orbits. Unfortunately, in most cases these truncation schemes are themselves so exceedingly complex that, even if lengthy calculations reproduce the empirical situation, the results offer little physical insight into the basic nuclear structure.

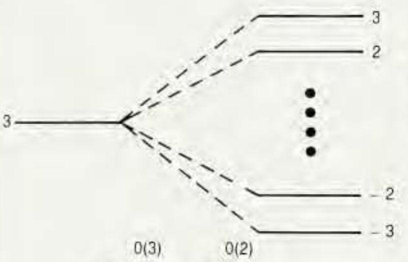
Interacting-boson model. This is where a new model[4] proposed by Arima and Iachello and known as the interacting-boson model enters. On the one hand, its basic assumptions place it in the realm of truncation models. On the other hand, its simplicity and its deep connection to geometry[8] confer on it the attractive features of collective models. Many aspects of this model and its extension to odd–even nuclei—including geometrical interpretations, applications to specific nuclei and microscopic shell-model formulations—are exhaustively discussed in the proceedings[9] of three major international conferences devoted to the subjects, and in a number of original papers.[10]

The basic assumptions[4] of the interacting-boson model are:

▶ Only valence nucleons are considered.

▶ Most of the possible configurations that the valence nucleons can form are ignored. In the model, the valence neutrons behave in such a way that, although they are fermions, they can be treated in pairs as effective bosons. The protons likewise form proton bosons. These bosons can have two "states" with angular momentum 0 ("s" bosons) or 2 ("d" bosons). The s bosons are analogous to Cooper pairs and the d bosons can be viewed as a generalization of this analogy for nonzero angular momenta. Note that this assumption is an enormous truncation of the shell model. Its adequacy is judged by the successes and failures of

Concept of Dynamical Symmetry



$E = C_1 l(l+1)\hbar^2 + C_2 m\hbar$

Eigenvalues of the angular momentum algebra O(3) and its degeneracy breaking decomposition into O(2) in the presence of a magnetic field, as, for example, in the Zeeman effect, in which atomic energy levels are split.

Dynamical symmetry is widely used in various branches of physics (for example, elementary-particle and atomic physics). Although the use of dynamical symmetry may involve complicated mathematical manipulations, its concept is simple and straightforward as can be seen from the following example. The more sophisticated applications are mere extensions of precisely the same concept.

According to quantum mechanics, there are three angular momentum operators, L_x, L_y, and L_z, which satisfy the "commutation relations"

$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

where the operator symbol $[A, B]$ means $AB - BA$. From these three operators, one may define another operator $\mathbf{L}\cdot\mathbf{L}$ $(= L_x^2 + L_y^2 + L_z^2)$, which possesses the important property of commuting with L_x, L_y, and L_z separately. This means that the angular momentum eigenstate $|l, m\rangle$ is simultaneously an eigenstate for the operators $\mathbf{L}\cdot\mathbf{L}$ and one of the three angular momentum operators. Generally, L_z is chosen for convenience. The quantum numbers l and m are the usual angular momentum quantum number and its z component, with l a positive integer and m equaling $-l, \ldots +l$. The angular representation of such a state is the well-known spherical harmonic $Y_{lm}(\theta, \phi)$.

The theory of angular momentum may also be stated in the language of dynamical symmetry based on a more abstract theoretical scheme. The mathematical underpinning is the concept of an "algebra," which is defined by its "basis" or "generators": a set of operators that satisfy appropriate commutation relations such as those for the operators L_x, L_y and L_z. From the generators, one may construct a so-called Casimir operator such as $\mathbf{L}\cdot\mathbf{L}$ that commutes with all the operators in the basis. The next task is to construct the so-called irreducible representations of the algebra. This is carried out with the concept of algebraic chains (or "chains," for short). Roughly speaking, this corresponds to finding a (sub)algebra whose basis is a subset of the original basis. The standard notation of "contain" is used to represent a chain: $G_0 \supset G_1$ where G_0 denotes an algebra and G_1 its subalgebra. Once the chain is constructed, there exist "standard" algebraic methods to construct the irreducible representations. Thus in this language, the original algebra of angular momentum is the so-called orthogonal algebra in three dimensions, O(3). The basis of the O(3) algebra consists of L_x, L_y, and L_z and the Casimir operator $\mathbf{L}\cdot\mathbf{L}$; the subalgebra is O(2) whose basis (with only one operator) is L_z. Thus we have the chain O(3)⊃O(2). The basis of the irreducible representations are $|l, m\rangle$ with l a positive integer and m equaling $-l, \ldots +l$. In effect, dynamical symmetry means that the Hamiltonian of a system is constructed from the Casimir operators of a chain. In the case of O(3), the simplest Hamiltonian is

$$H = C_1\mathbf{L}\cdot\mathbf{L} + C_2 L_z$$

This Hamiltonian exhibits a crucial property of chains for nuclear physics (see the adjoining figure), namely that each step breaks some of the degeneracy, distinguishing formerly degenerate levels according to an additional quantum number. In the present example, the eigenvalues for the Hamiltonian given above are $C_1 l(l+1)\hbar^2 + C_2 m\hbar$. Clearly, if C_2 vanishes, the system is "rotationally invariant" and has for each l a $(2l+1)$-fold degeneracy. If C_2 differs from 0 (for example, if an external magnetic field is switched on), that is, if the O(3) symmetry is broken by the O(2) symmetry, then the degeneracy is lifted.

the model and by microscopic shell-model calculations themselves.

▶ The model allows for simple interactions between these bosons—or nuclear pairs—in which, for example, an s boson is changed into a d boson or two d bosons into two s bosons. The number of bosons N_B characterizing a nucleus is equal to the sum of the number of s bosons (n_s) and the number of d bosons (n_d) and is given by half the number of valence nucleons. This number is constant for a given nucleus.

Calculations in the interacting-boson model can be carried out by the brute-force method of directly solving the Schrödinger equation defined by the basic degrees of freedom; that is, by diagonalizing the Hamiltonian to yield eigenvalues, eigenfunctions, transition rates and the like. This Hamiltonian contains several parameters that can be chosen either by fitting them to empirical data or, ideally, by reference to microscopic shell-model calculations. It involves only simple interactions between the s and d bosons. A slightly simplified form of this Hamiltonian is given by

$$H = \epsilon n_d - \kappa \mathbf{Q} \cdot \mathbf{Q} - \kappa' \mathbf{L} \cdot \mathbf{L} + \kappa'' \mathbf{P} \cdot \mathbf{P}$$

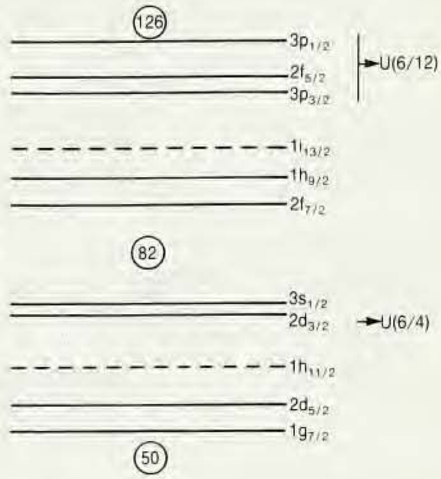
where n_d is the number of d bosons, $\mathbf{L}$ is the boson angular momentum operator and $\mathbf{Q}$ and $\mathbf{P}$ are boson operators that, roughly speaking, correspond to quadrupole and pairing interactions between bosons. In practice these operators are written in terms of the "creation" and "annihilation" operators discussed below. To solve this model, one simply chooses appropriate values for the parameters describing each interaction term and diagonalizes the Hamiltonian.

In many cases, however, it is better to use an alternate approach that exploits the fact that the algebraic structure of the interacting boson model leads naturally to the concept of dynamical symmetry. This concept is illustrated in the box on page 28; its use in the interacting-boson model is more complex but is a direct generalization of the same ideas. This approach to collective states, schematically illustrated on the top left in figure 3, only works well for certain very special nuclei, but is of more general interest because it provides a framework of ideas from symmetry or geometry by which one can understand the calculations for much broader classes of nuclei. To pursue this idea of dynamical symmetry in the interacting-boson model, it is convenient to represent the s and d bosons by the so-called boson "creation" and "annihilation" operators $s^\dagger$, s and $d^\dagger_\mu$, d_μ (μ corresponds to the azimuthal quantum number, with values -2, -1, 0, 1, 2). Then the 36 operator products of the form $s^\dagger s$, $d^\dagger_\mu d$, $sd^\dagger_\mu d$, and $s^\dagger d_\mu$ satisfy the appropriate commutation relations (see the box on page 28) to form the basis of a six-dimensional algebra called U(6). For this system then, the highest symmetry is U(6) and the lowest symmetry must be O(3) (rotational invariance). In the language of algebraic chains analogous to those of the O(3) algebra discussed in the box on page 28, this means that the U(6) algebra (symmetry) is broken by subalgebras (subsymmetries) until the O(3) algebra (symmetry) is reached. There are three such chains:

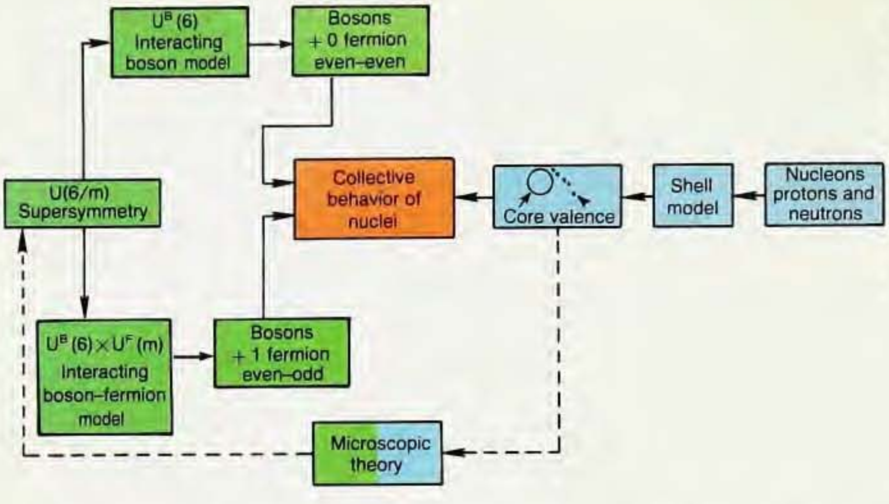
$$\mathrm{U}(6)\begin{cases}\rightarrow \mathrm{SU}(3)\supset \mathrm{O}(3) & \kappa\\ \rightarrow \mathrm{O}(6)\supset \mathrm{O}(5)\supset \mathrm{O}(3) & \kappa''\\ \rightarrow \mathrm{U}(5)\supset \mathrm{O}(5)\supset \mathrm{O}(3) & \epsilon\end{cases}$$

Each chain results when all but one of the terms in the Hamiltonian given above vanishes. The relevant coefficient in H for each chain is indicated. As in the angular-momentum case discussed earlier, each step in a given chain breaks a degeneracy, introduces a new quantum number or a set of new quantum numbers to distinguish former degenerate levels and introduces a corresponding term in the eigenvalue equation. Each step provides new selection rules on transitions and gives analytical expressions for allowed transitions. This concept of successive degeneracy breaking is illustrated in figure 4 for the O(6) chain and has been discussed[9] by Itzhak Bars. An important point is that only the levels on the far right can be observed in a real nucleus. Thus it is only by detailed spectroscopy of these levels and transitions among them that one can hope to glimpse evidence for or against the validity of the set of quantum numbers, selection rules and so on, that characterize a given chain and provide evidence for the parent symmetry on the far left. Each chain represents a type of nucleus of a specific structure and has a specific geometrical interpretation. It is beyond the scope of this article to show this, but it is important—and helpful for understanding the interacting-boson model—to give the result. The SU(3) symmetry corresponds to a non-spherical (or "deformed") nucleus shaped like a football that can both vibrate and rotate. The U(5) symmetry depicts a spherical nucleus and its vibrational excited modes. The O(6) symmetry also describes a deformed nucleus, but one that is not axially symmetric—it can be crudely visualized as a "squashed" football although the amount of squashing is not constant in time.

A convenient way to illustrate the three symmetries is with the triangle shown in the lower right of figure 1. Each vertex represents one of the symmetry-breaking chains and, therefore, a particular nuclear shape. The legs of the triangle represent particular cases of transition regions in which, over a series of nuclei neighboring in mass (for example, the even–even osium and platinum nuclei), the structure undergoes a transition from one symmetry to another. Such transition regions, though extremely difficult to explain in traditional models, are the epitome of simplicity in the interacting-boson model. The character of any nucleus in such a region can be specified by a single parameter reflecting its relative position along the leg. Of course many nuclei are more complex still, corresponding to internal positions in the triangle, and these require more parameters for their calculation. The energy levels of these nuclei exhibit, to some degree, the features characteristic of two or three different symmetries. Nevertheless it is remarkable that most of the best-studied transition regions correspond closely to one of the three legs. Various nuclei exhibiting[10] either a pure symmetry or a simple mixture of symmetries are labeled next



Portion of the energy levels of the nuclear shell model. Each level is labelled by quantum numbers, n, l and j; n is the principal—or shell—quantum number, l is the orbital angular momentum, indicated by letters according to the usual notation in atomic physics (s, p, d, and so forth), and j is the total angular momentum given by the vector sum of l and the intrinsic nuclear spin of ½. The circled numbers are the "magic" numbers of nucleons (either protons or neutrons) corresponding to a closed shell. The orbits corresponding to the U(6/4) and U(6/12) supersymmetries discussed in the text are indicated. Figure 2



Interrelationships between nuclear models of collective states. The shell model approach is shown on the right and leads to the interpretation of collective states in terms of the detailed (microscopic) nature of the valence nucleons. The supersymmetry approach to both even–even and odd–even nuclei is on the left. The box at the bottom suggests a hoped-for link in which microscopic shell model calculations would predict the parameters used in phenomenological supersymmetry fits to real nuclei. Figure 3

to the triangle. The triangle incorporates a color code for the symmetries and for intermediate situations. This code is then used to classify all the even–even nuclei in the rare earth–Os–Pt region as shown in the main part of figure 1.

The prediction[4] of the three subalgebras, denoted U(5), SU(3) and O(6), proved to be a major event in the early emergence of the interacting-boson model since nuclei resembling the geometrical counterparts of the first two of these symmetries were already well known while the O(6) limit was not. Its empirical discovery (see the first article of reference 10) in Pt^{196} played a major role in establishing the model as an important new approach. Indeed, the example of Pt^{196} nicely illustrates one aspect of the importance and power of symmetry ideas. The level scheme of Pt^{196} appears at first glance as an incredibly complicated jumble of levels and γ-ray transitions. It is only upon comparison with the level patterns of the symmetry that the striking simplicity inherent in the empirical level scheme suddenly becomes manifest.

Clearly the inherent assumptions of the interacting-boson model render it applicable only to the collective states of even–even nuclei, which below a certain excitation energy are described as a finite interacting system of bosons. Above this excitation energy there is sufficient energy to excite individual nucleons to different orbits, thereby breaking the system of paired-off nucleons that constitute the s and d bosons. Such states are outside the scope of the interacting-boson model. Also outside the scope of the model are the low-lying states of an odd–even nucleus, which, of course, must have one unpaired nucleon (fermion). However, such nuclei can be treated within the framework of an extended version of the model, called the interacting boson–fermion model which was proposed[11] five years ago by Iachello and Olaf Scholten. Here the Hamiltonian has the plausible tripartite form

$$H_{\rm IBFM} = H_{\rm IBM} + H_{\rm SP} + H_{\rm INT}$$

where $H_{\rm IBM}$ is the nuclear Hamiltonian that describes the collective degrees of freedom and which, in this model, is that used for the neighboring even–even nucleus; $H_{\rm SP}$ is a single-particle term reflecting degrees of freedom of the odd fermion; and the most important term, $H_{\rm INT}$, is an interaction term specifying the coupling of the unpaired particle and the collective degrees of freedom. The form of the Hamiltonian (although not the structure of the individual terms) is the same as that describing the coupling of electrons and phonons in a metal. Some of the original calculations of the interacting boson–fermion model for odd–even nuclei were carried out by the brute-force method of diagonalization of this Hamiltonian. However, there are inherent difficulties with such calculations. The most significant of these is that, without model-dependent guidance from microscopic shell-model calculations to simplify the problem, the number of parameters in such a Hamiltonian for the boson–fermion model is approximately the cube of the number of single-particle levels. This makes a meaningful physical interpretation very difficult indeed.

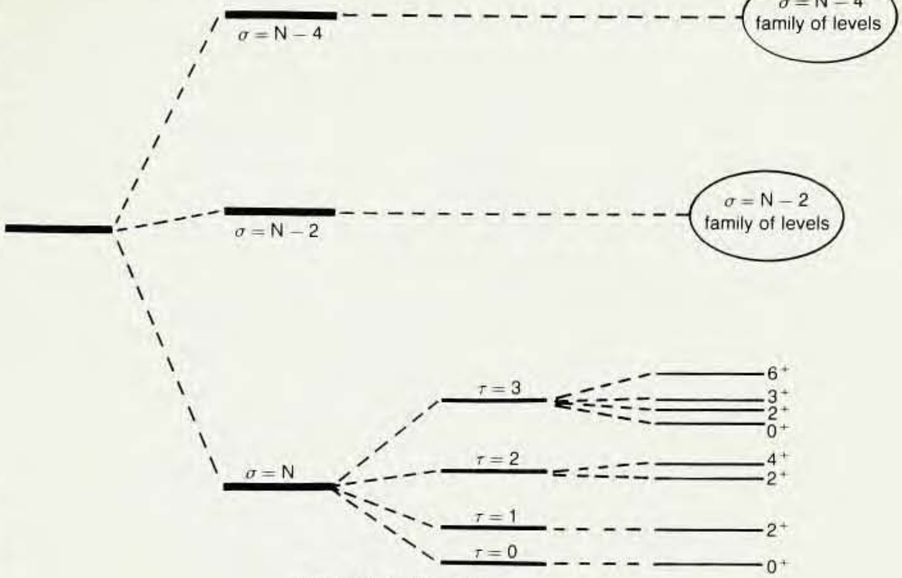
Supersymmetry concepts for nuclei

An alternative approach is to explore the dynamical symmetries in the problem in the same way as in the interacting-boson model. As with the boson-model, these symmetries for the odd–even nucleus are in effect a model for the crucial parameters of the Hamiltonian. Of course the situation here is more complicated than in the interacting-boson model because of the additional fermionic degrees of freedom in the system, and its exploitation has only recently been achieved by the introduction of supersymmetry ideas as described below. As we discuss in the box on page 33, the fermion degree of freedom can be expressed in an analogous fashion to the boson degree of freedom in terms of an algebraic structure as a fermionic algebra $U^F(m)$ where m is the total degeneracy of the single particle levels. Thus the highest symmetry of such a boson–fermion system is the product algebra $U^B(6) \times U^F(m)$. Unlike the interacting-boson model, which has only three sequences of subalgebras, or chains, the number of possible chains for the boson–fermion model are in general more numerous. A further complication (see figure 2) is that the level scheme that corresponds to a single particle changes from one region of the periodic table to another. The value of m changes accordingly; thus, for the odd nucleus, there is no "universal" highest symmetry such as U(6) in the boson model. For example, the appropriate single-particle levels for the odd-parity states for the platinum isotopes are $p_{3/2}$, $p_{1/2}$, and $f_{5/2}$. This means that the highest symmetry is $U^B(6) \times U^F(12)$. There are many ways to break such symmetries to reach the so-called Spin (3) algebra (see the box on page 33), the analogous algebra in the odd–even nucleus to O(3) for rotational invariance for even–even nuclei. An example (not unique) starting with, say $U^B(6) \times U^F(12)$, is a chain of the form

$$\begin{aligned} &U^B(6) \times U^F(12) \supset U^B(6) \times U^F(6) \times U^F(2) \\ &\quad \supset O^B(6) \times O^F(6) \times SU^F(2) \\ &\quad \supset O^{B+F}(6) \times SU^F(2) \\ &\quad \supset O^{B+F}(5) \times SU^F(2) \\ &\quad \supset O^{B+F}(3) \times SU^F(2) \supset \mathrm{Spin}(3). \end{aligned}$$

Each step corresponds to a further breakup of the degeneracy of the levels and introduces a new quantum number to classify them. This is exactly analogous to the "chain decompositions" of the familiar angular-momentum algebra (see the figure in the box on page 28) and the U(6) algebra of the interacting-boson model that are shown in figure 4. A great deal of mathematical technology now exists to carry out such

Calculated energy levels of an atomic nucleus, for example Pt, supplied by the U(6) algebra of the interacting-boson model. The successive degeneracy breakings and quantum numbers are indicated and the eigenvalue expression given at the bottom. *A*, *B*, and *C* are adjustable parameters. Figure 4



$$U(6) \supset O(6) \supset O(5) \supset O(3)$$

$$E = E_0 + A\sigma(\sigma + 4) + B\tau(\tau + 3) + CJ(J + 1)$$

chain decompositions.

Clearly the bosonic description for the nucleus in either the boson or the boson–fermion model can only be an approximate one, because real nuclei consist solely of fermions (protons and neutrons). Therefore the $U^B(6)$ symmetry for even–even nuclei or the $U^B(6) \times U^F(m)$ symmetry for odd–even nuclei cannot be the highest symmetry for the system. In a sense, these are only "effective" symmetries. This is quite different from the previous studies of dynamical symmetries in nuclear physics where one usually begins with the highest symmetry within the model space. For example, the seminal work[12] of J. Phillip Elliott on the SU(3) model begins with the $U^F(12)$ symmetry, which is the highest symmetry for an $s_{1/2}$, $d_{3/2}$, $d_{5/2}$ model space.

Once the symmetry—whether boson or boson–fermion—is recognized only as effective, an intriguing possibility can be raised, namely whether its symmetry, $U^B(6)$ or $U^B(6) \times U^F(m)$ respectively, could conceivably serve the role of a symmetry-breaking mechanism of a still higher symmetry, just as O(6) breaks the U(6) symmetry in the even nucleus or O(2) breaks the O(3) angular-momentum symmetry. Iachello[3] postulated in 1980 that the concept of supersymmetry is such a (higher) symmetry:

$$\text{supersymmetry} \begin{cases} \rightarrow \text{interacting-boson model symmetry} \\ \rightarrow \text{interacting boson–fermion model symmetry} \end{cases}$$

For example, the above chain modified to incorporate dynamical supersymmetry as the highest symmetry must now be altered to begin with the supersymmetry algebra U(6/12):

$$U(6/12) \supset U^B(6) \times U^F(12) \supset \ldots$$
$$\supset \text{Spin}(3).$$

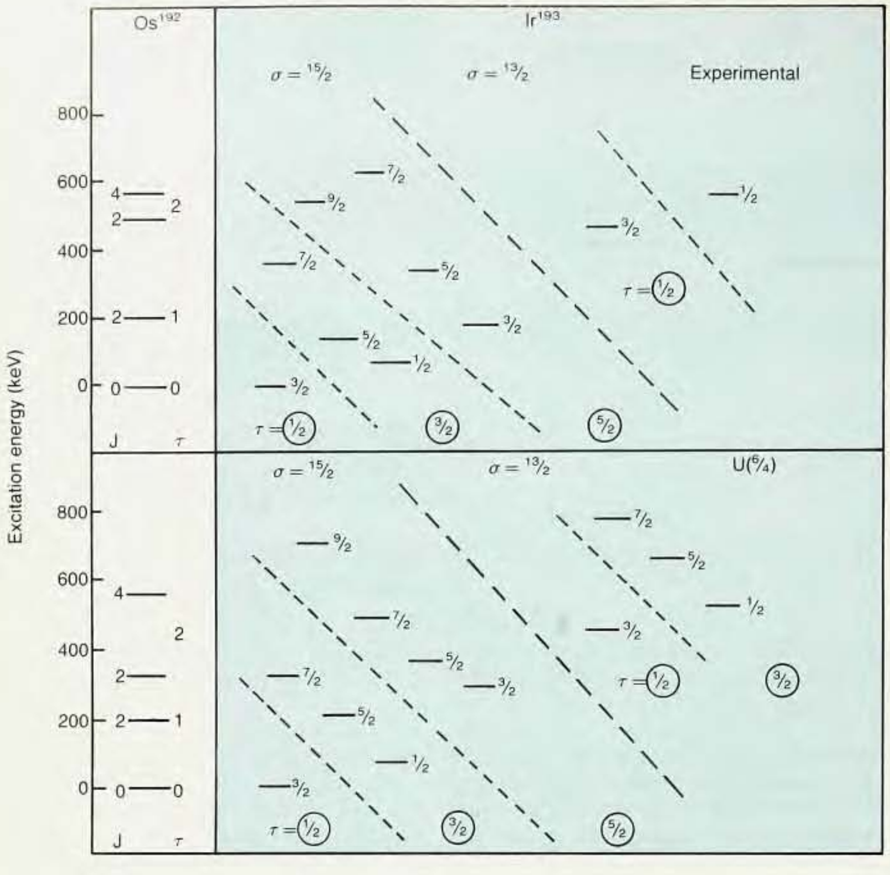
Such a generalized approach to collective states is included in figure 3.

Having postulated dynamical supersymmetry as the unifying symmetry for the even–even and odd–even systems, it would be interesting to know whether such a symmetry can be detected experimentally. There are two levels to this question. The first is: Given a complete chain with a superalgebra $U(n/m)$ as its highest symmetry, can one find a set (or sets) of data that are well-predicted by the theory? The second, deeper level is, how might one demonstrate the existence of such a symmetry? The second question is far more difficult to address. Ideally, one would like to have experimental results that explicitly point to—and require—the highest symmetry of the chain. If this could not be achieved, and if, at least, one could establish evidence for the existence of each of the subalgebra chains, this would point "upstream" to the superalgebra as their "source." To do this, needless to say, is an arduous task, both theoretically and experimentally, and therefore most tests of supersymmetry to date have attempted to answer the first question by comparing a given chain with a specific set of data. In line with this, we hope that with the accumulation of more empirical evidence it will be possible to answer the second question as well.

We shall now discuss the structure of the two most studied of these chains in some detail and summarize the experimental situation. As with symmetries in the interacting-boson model, each such chain decomposition of a superalgebra corresponds to a specific choice of the coefficients in the boson–fermion model Hamiltonian. There is no *a priori* reason to expect that any given nucleus will be described by that choice. However one can hope that, in nuclear regions where the even–even nuclei display one of the symmetries of the interacting-boson models, conditions may be ripe for an even–even and odd–even pair of nuclei to be characterized by an appropriate supersymmetry chain.

Experimental tests

Figure 2 shows a number of single-particle energy levels for an unpaired nucleon. This translates into a rich variety of possible supersymmetry schemes because the available single-particle levels determine the size of the fermion sector (that is, m) in the superalgebra $U(6/m)$. Two cases have been rather thoroughly studied (but many others can be, and are being, looked into): supersymmetries based on the groups U(6/4) and U(6/12). (See references 3 and 13 for the original theoretical papers, references 14–16 for further embellishments and references 17 and 18 for experimental tests of supersymmetry in nuclei. References 1 and 19 provide recent summaries of the status of the field.) The first supersymmetry studied, U(6/4), corresponds to a fermion in a single orbit with $j = 3/2$. The other, U(6/12), corresponds to a fermion that can be in any of the three orbits corresponding to j having the value $1/2$, $3/2$ or $5/2$. The interest in these two cases stems directly from the fact that the even–even Pt isotopes near $A = 196$ ($Z = 78$, $N = 118$) are a very good manifestation of a dynamical symmetry in the boson model, namely the chain including the O(6) symmetry, as is indicated by red color describing these nuclei in figure 1. Figure 2 shows that the positive-parity proton orbits in this region are the $s_{1/2}$ and $d_{3/2}$ orbits



Measured levels of the supersymmetry pair Os^{192} and Ir^{193} compared with those predicted by the U(6/4) symmetry. The σ and τ quantum numbers are given, as are the level angular momenta. (See references 3, 13 and 17.) Figure 5

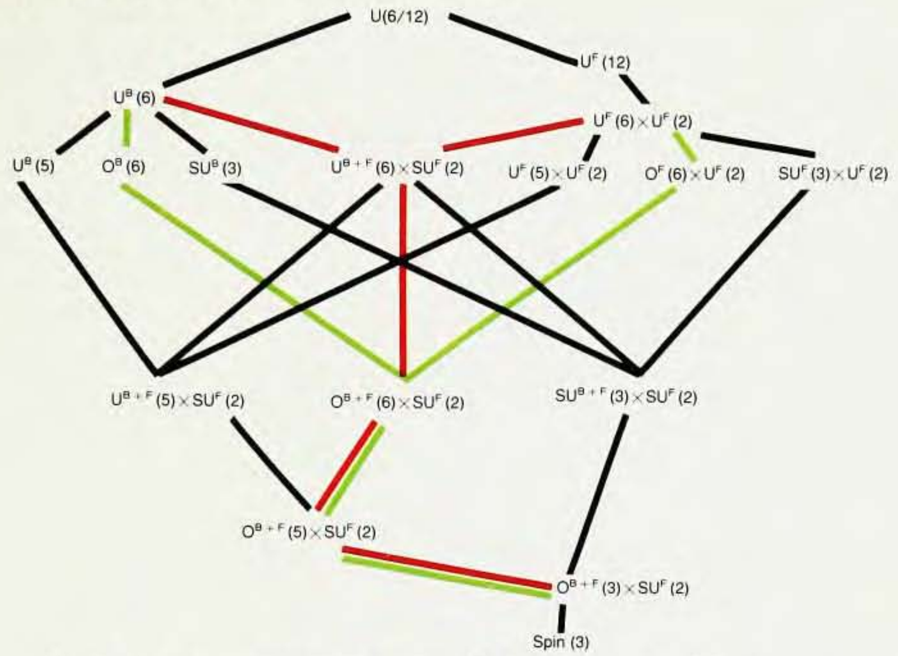
while the negative-parity neutron orbits are the $p_{1/2}$, $p_{3/2}$ and $f_{5/2}$ orbits. Thus a U(6/12) supersymmetry should describe this latter case—that is, odd–even and even–even platinum isotopes—while U(6/4) may be tested as a partial description (accounting for the $d_{3/2}$ orbit and neglecting the $s_{1/2}$ orbit) of the odd–even iridium ($Z = 77$) nuclei and the even–even osmium ($Z = 76$) nuclei. The U(6/4) scheme, as the first supersymmetry to be proposed for nuclei, will be discussed first. The first priority in testing it is to isolate and identify those levels in the odd–even nucleus that have an unpaired proton in the $d_{3/2}$ orbit. Once this subset of levels is distinguished from that in which the $s_{1/2}$ orbit is occupied, one can use a variety of experimental techniques to study the structure of the appropriate levels.

Thus, for a starting point, one needs an experimental probe that is orbit-sensitive. A good example is the (He^4, H^3)—or (α, t)—reaction, in which an incident α particle approaches a target nucleus. Once within the range of the nuclear potential, a proton can be extracted from the α particle, which becomes a triton. The proton is transferred to the target nucleus (of mass A) and circumnavigates the residual ($A + 1$) nucleus in any one of the available low-lying shell-model orbits. Only those states in the final nucleus will be formed that look like the target nucleus plus one odd nucleon in a specific orbit.

In actual tests of the U(6/4) scheme, the inverse reaction, (t, α), has been used on a target of Pt^{194} to study levels in Ir^{193} by Jolie Cizewski and coworkers[17] at the tandem accelerator of Los Alamos National Laboratory. A comparison[3,13,17] of empirical and supersymmetry-predicted levels in the even–even and odd–even pair of nuclei, Os^{192}–Ir^{193}, is shown in figure 5 where the levels are labelled by the three quantum numbers J, for the angular momentum of each state, σ, which is a major quantum number distinguishing families (representations) of levels and τ, a quantum number distinguishing levels within a family. (See the discussion in the box on page 33.) The agreement is impressive indeed. The angular momenta predicted and observed are the same, and the level patterns are nearly identical—in particular the divisions according to the τ and σ quantum numbers. Moreover, the relative cross sections in the (t, α) reaction making Ir^{193} obey the selection rules and ratios expected in this supersymmetry. There are discrepancies in details, but the overall agreement is encouraging. Other studies[17] have investigated γ-ray transitions in this region and these also show reasonable accord with the U(6/4) scheme. Tests[17] of the U(6/4) scheme for the Au^{197}–Pt^{196} pair of nuclei by Jacques Vervier and his coworkers at the cyclotron of Louvain-la-Neuve, Belgium, also show good agreement for γ-ray transitions and reasonable agreement for the nuclear reaction Pt^{196}(He^3, H^2)Au^{197}. On the other hand, other studies[17] by a group led by Michel Vergnes at the synchrotron at Orsay, France, reveal sizeable descrepancies with the U(6/4) scheme in particle-transfer reactions of the type (He^3, H^2) leading to Pt^{194} and (H^2, He^3) leading to Pt^{196}. Nevertheless, all the existing data taken together have provided the first empirical evidence for supersymmetry in nuclei while, at the same time, they have disclosed a degree of U(6/4)–O(6) chain breaking, especially with increasing mass. Cizewski has emphasized that this breaking is at least partially related to the restriction to an incomplete fermion space, pointing to the need for a multi-j scheme.

Recently, such a scheme based on the U(6/12) group has, in fact, been worked out.[13] As noted above, this supersymmetry corresponds to a fermion that can occupy orbits where j equals $\frac{1}{2}$, $\frac{3}{2}$ or $\frac{5}{2}$ and should apply to the Pt isotopes. It is crucial to note that in this case the supersymmetry should predict all low-lying, low-spin, negative-parity levels, so one needs an experimental probe that is not selective. The radiative neutron capture, or (n, γ), reaction is ideal. As contrasted with the direct reactions discussed

Supersymmetry concepts in nuclei



Dynamical supersymmetry is the dynamical symmetry proposed for a mixed fermion and boson system. It provides a framework that allows one to treat systems (nuclei, polymers, elementary particles or whatever) in which there are both bosons and fermions in such a way that both bosonic and fermionic aspects are on an equal footing and yet their individuality also emerges. This is a radical departure from all symmetries used hitherto in physics, because it must unite the concepts of fermions (which carry intrinsic half-integer spins, satisfy Fermi–Dirac statistics and obey the Pauli exclusion principle) and bosons (which carry integer spins, satisfy Bose–Einstein statistics and obey no exclusion principle). In the nuclear case, supersymmetry thus provides a way to unify the understanding of even–even and odd–even nuclei, and, moreover, does so in a way that highlights the intuitively appealing (common) symmetry aspects (geometrical shape concept) of the nuclei involved. Supersymmetry in nuclei is analogous to, and is an extension of, the normal dynamical symmetry of the boson model for even–even nuclei and indeed, when applied to such nuclei, yields the familiar interacting-boson results. We have learned from the interacting-boson model that, if paired fermions in a system behave as bosons, the system also possesses the dynamical symmetry U(6). Likewise, one has known from Gino Racah's early work in atomic spectroscopy that a system of fermions moving in a field possesses the dynamical symmetry U(m) where m is the total (degenerate) number of single particle orbits of the system; that is, m equals $\Sigma_\alpha (2j_\alpha + 1)$ where the j_α are the angular momenta of the shell model orbits available to the fermion. Thus a combined system of bosons and fermions will be described by $U^B(6) \times U^F(m)$. This is not yet a supersymmetry since the boson and fermion "sectors" are still distinguishable. However, if the boson and fermion systems are mixed, such that bosons may turn into fermions and vice versa, then we have a dynamical supersymmetry system, U(6/m). The mathematics of such superalgebras originated only about a decade ago (whereas the Lie algebra of regular symmetries was developed over a hundred years ago); it is still very much a topic of current research in mathematical physics.

As mentioned in the text, two main examples of supersymmetries have been proposed for nuclei. They differ in the allowed orbits that the odd fermion can occupy. The first test of dynamical supersymmetry in nuclear physics, proposed by Baha Balantekin, Bars and Iachello,[13] involved a fermion with j equaling $\frac{3}{2}$, coupled to the $O^B(6)$ even–even core nucleus described well by the interaction-boson model. Thus the highest symmetry is the "single-j" supersymmetry U(6/4), and the chain is

$$\begin{aligned} U(6/4) &\supset U^B(6) \times U^F(4) \\ &\supset O^B(6) \times SU^F(4) \supset O^{B+F}(6) \\ &\supset O^{B+F}(5) \supset O^{B+F}(3) \end{aligned}$$

In line with the concept of dynamical symmetry outlined in the box on page 28, each step here involves a greater degree of degeneracy breaking and introduces a new quantum number: for example, σ for the $O^{B+F}(6)$ step and τ for the $O^{B+F}(5)$ link.

The second supersymmetry to be studied[13] is a "multi-j," in which the superalgebra is U(6/12), and where therefore corresponds to the unpaired nucleon in the $p_{1/2}$, $p_{3/2}$, $f_{5/2}$ orbitals. In this case, the chain decomposition is nontrivial. It involves the concept of the "pseudo-orbital" method whose technical aspects are beyond the scope of this article. The entire chain decomposition is also more complex than the above because there are many "paths." It can be best illustrated by a "tree" diagram as given in the adjoining figure. The figure includes chains involving all three of the even symmetries of the interacting-boson approximation: $U^B(5)$, $SU^B(3)$ and $O^B(6)$. In addition, the tree shows two types of coupling depending on whether the boson and fermion parts are combined at the level of $U^{B+F}(6)$ or lower in the chain at $U^{B+F}(5)$, $SU^{B+F}(3)$ or $O^{B+F}(6)$. For the decompositions involving $O^B(6)$, which are the ones that have been most tested to date, the two chains are drawn in red and green, respectively. To illustrate the use of the figure, the green chain in an $O^B(6)$ decomposition, reads

$$\begin{aligned} U(6/12) &\supset U^B(6) \times U^F(12) \\ &\supset U^B(6) \times U^F(6) \times U^F(2) \\ &\supset O^B(6) \times O^F(6) \times SU^F(2) \\ &\supset O^{B+F}(6) \times SU^F(2) \\ &\supset O^{B+F}(5) \times SU^F(2) \\ &\supset O^{B+F}(3) \times SU^F(2) \supset Spin(3) \end{aligned}$$

Again, each step involves a further breaking of the degeneracy and a new quantum number: for example, (σ_1, σ_2) for the $O^B(6) \times O^F(6) \times SU^F(2)$ step and (τ_1, τ_2) for the $O^{B+F}(5) \times SU^F(2)$ step.

above, this is a compound nuclear reaction is which the incident neutron is absorbed by the target (mass A) nucleus; the result is an isotope of mass $A + 1$ in a highly excited state with an energy approximately equal to the neutron binding energy. This state is extraordinarily complex, its wave function containing amplitudes for many different multiparticle configurations. Owing to this complexity, the intensity of the γ-rays emitted when the state decays to lower-lying levels follows a statistical law. By carrying out (n, γ) experiments with non-monoenergetic neutrons, one can average out the statistical fluctuations: then, all states of a given spin will be populated equally (except for a known energy dependence). A corollary of this is the assurance that the complete set of such states up to some excitation energy will be disclosed. The concept of spectroscopic "completeness"[20] is a new one in nuclear physics and its exploitation in the (n, γ) reaction has played an important role in revitalizing interest in this long-familiar tool, just as the interacting-boson model has led to a renaissance of interest in nuclear structure itself. This (n, γ) averaging technique, known as average resonance capture, is carried out at Brookhaven National Laboratory at the High Flux Beam Reactor which provides a high-intensity flux of neutrons ranging in energy from thermal (much less than 1 eV) to many keV. If a beam of these neutrons is passed through special filter materials of natural Sc or Fe^{56}, a neutron beam is produced with mean energy of 2 or 24 keV respectively, which is ideal for average resonance capture spectroscopy. In addition, it is important to carry out detailed studies of individual γ rays connecting low-lying levels. These can best be studied using the high-precision bent-crystal spectrometers at the Institut Laue–Langevin in Grenoble, France. These detectors use the principle of Bragg diffraction in which the deexcitation γ rays to be studied are scattered at angles that are a sensitive function of their energy.

The combination of these techniques has been used[18] by David D. Warner and coworkers to test the U(6/12) scheme in Pt^{195}, and average resonance capture spectroscopy alone has been carried out for Pt^{197} and Pt^{199}. The Pt^{195} nucleus is the best studied. There are two particular decompositions of the U(6/12) superalgebra that incorporate O(6) symmetries and might be considered for comparison with the data. These schemes are cited in the box on page 33 and denoted in the adjoining figure in green and red. It turns out that the red chain[13] reproduces the empirical results, especially the energy levels, better. It differs from the green chain primarily in the amount of interaction between the unpaired nucleon and the underlying even–even core nucleus that this nucleon orbits. This "coupling" is greater in the red chain and appears to be a better depiction of the data. It is compared with the data for Pt^{195} in figure 6: It is clear that, overall, there is excellent agreement with experiment. Most importantly, there is a one-to-one correspondence of observed and predicted levels with the proper spins, and the energy agreement is good. This achievement is not easily claimed by other nuclear models for this region. The only significant discrepancy is that the theoretically predicted energy levels of the second representation ($[N_1, N_2] = [N, 1]$) are too compressed. Recently, Hong-Zhou Sun, Michel Vallieres and their coworkers proposed[14] an extension of the U(6/12) scheme, incorporating higher-order terms (Casimir operators) in the Hamiltonian in an effort to eradicate this discrepancy; although this idea is interesting, futher testing in other mass regions is needed for a better assessment of its usefulness. It is significant that the same parameters that produce the results in figure 6 also give reasonable agreement for the low-lying levels of the even–even nucleus, Pt^{196}. This fact is important, for it points rather clearly to the existence of the parent super-algebra U(6/12), as opposed to "decoupled" $U^B(6)$ and $U^F(12)$ algebras, as the highest symmetry and, as such, addresses the supersymmetry issue at the second level noted above. These results inspired subsequent studies of the U(6/12) supersymmetry. The resonance-capture research[18] on Pt^{197} and Pt^{199} suggested the continuity of a supersymmetry interpretation although other work[17] by Vergnes' group, using selective transfer reactions such as H^2, p) and (p, H^2), indicates a gradual breakdown of the U(6/12) scheme with increasing mass. To establish more definitively the applicability and evolution with mass of the U(6/12) supersymmetry in this region, it will be important to study γ-ray transitions in these nuclei and to compare their absolute and relative strengths with the supersymmetry. Work along these lines is also in progress.

To summarize, there is now growing empirical evidence that the concept of a U(6/12) supersymmetry in a chain involving the O(6) symmetry has utility for odd nuclei near Pt^{196} and presumably reflects at least the approximate validity of the underlying symmetry concepts.

Very recently, the U(6/12) scheme—this time in a chain incorporating the SU(3) symmetry of the boson approximation—has been tested[18] for W^{185} by Warner and coworkers at Brookhaven. Again the problem of compression of levels occurs; in this case most clearly in comparing fits for the even–even and odd–even nuclei in the higher representations. Nevertheless, the predictions for the odd–even nucleus W^{185} reproduce the empirical data very well: In particular they seem to account for some levels around 600 keV of excitation energy that were puzzling in previous models of these nuclei. The SU(3)-based supersymmetry chain naturally predicts their presence and, indeed, may account for their detailed structure. These results, when combined with the Pt results discussed above, go a long way towards supporting an affirmative answer for our first question: Given a superalgebra—in this case based on U(6/12)—are there data that satisfy its predictions?

It should also be mentioned that the single-j supersymmetry U(6/4) discussed earlier has very recently been expanded[14] by Yin-Sheng Ling and collaborators in another multi-j supersymmetry scheme, U(6/20). In this scheme, the fermion can occupy not only the orbit with j equaling $3/2$, but orbits in which j equals $1/2$, $5/2$, and $7/2$ as well. It is interesting to note that some of the difficulties associated with the U(6/4) scheme are now removed by this extension.

The importance and appeal of supersymmetry ideas to nuclei is at least fourfold. First, they provide a much needed unification of our understanding of even–even and odd–even nuclei (and, potentially, of odd–odd species as well). Second, they offer insight into the inherent, underlying symmetry structure of atomic nuclei and are also a powerful tool for discerning the beauties of simplicity amidst the apparent complexity of detailed nuclear-level schemes. It is worth amplifying this in a historical context. When confronted by a seemingly complicated physical system, the physicist's quest is to seek a unifying—and thus, one hopes, simplifying—understanding. For example, once Mendeleyev recognized the pattern of the atomic elements in the form of a periodic table, it was inevitable that an understanding of the underlying structure of atoms would be forthcoming (even though it came about a half century later). Similarly, one may understand the dynamical symmetry in collective nuclear structure, discussed in this paper, in this light as well. The properties of the even–even nucleus (such as energy spectra or transition rates) at first glance are very complicated indeed. Yet the much-heralded geometrical picture provided a needed appreciation of the collective manifestations in this complicated system. This appreciation has led to the understanding, some twenty years later, of the underlying structure by means of dynamical symmetry. Today the properties of the even–odd nucleus by themselves also appear very complex. An interpretation of both the even–even and even–odd nuclei in a simple framework appears even more remote, although a unified understanding of these two physical systems has long been the goal of nuclear-structure physics. It is indeed remarkable that dynamical symmetry for the even–even nucleus provides a clue pointing us in a direction that may lead to this unified understanding, via dynamical supersymmetry. The third attractive aspect of supersymmetry schemes is that they provide simple analytic expressions for eigenvalues and transition rates that avoid the need for complex multiparameter diagonalization of the full Hamiltonian of both the boson and the boson–fermion models. Fourth, although ideas on supersymmetry are now widespread in physics and under intense study in many subfields, only in nuclear physics is there any substantial body of experimental evidence reflecting on their validity. Although the specific supersymmetry schemes appropriate to nuclei differ from their counterparts in, for example, high-energy or condensed-matter physics, the emerging consensus on their usefulness here confers on the broader field a new level of general interest and a strong motivation for increased activity. Clearly, however, it is still too early to make sweeping conclusions concerning the validity of nuclear supersymmetries in general. At the moment, the evidence for the level patterns they predict is encouraging, but discrepancies nevertheless remain. It will only be with the accumulation of considerably more data that a full picture of their utility can be painted. Even if it turns out that no nuclei truly follow the supersymmetry predictions, the concept can still be of immense value in providing a new touchstone. Just as in even–even nuclei, where the symmetry of the interacting–boson model led immediately to a new understanding of such regions as the Pt–Os and Sm–Gd nuclei in terms of systematic deviations from one or another of the symmetries of the boson models, the structure and properties of supersymmetries may provide a foundation for simplifying the interpretation of extensive sequences of even–even and odd–even nuclei.

* * *

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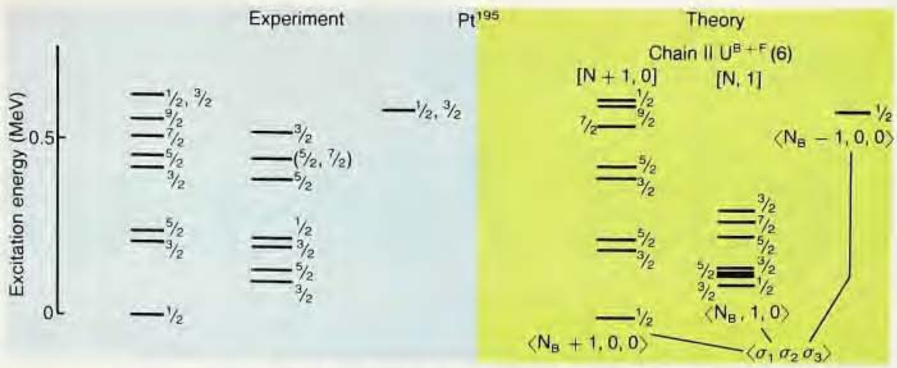

Comparison of the levels of Pt^{195} with those of the U(6/12) (red) chain. The fit shown is based on parameters used in reference 13. Several quantum numbers are shown that distinguish different states. The most important ones are $[N_1, N_2]$ which are the quantum numbers for $U^{B+F}(6)$ and $\langle \sigma_1, \sigma_2, \sigma_3\rangle$ which are the quantum numbers for $O^{B+F}(6)$. Figure 6

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