Large-N quantum mechanics and classical limits

Increasing the number of degrees of freedom surprisingly simplifies the analysis in many quantum theories, often making it possible to calculate physical observables.

Laurence G. Yaffe

For a physical theory to be useful, we should be able to extract from it quantitative predictions for physical observables. With most theories, this process of practical application requires some approximation method that is both tractable and adequately accurate. The lack of a useful approximation scheme can seriously impede progress in a field of research. Molecular quantum mechanics and critical phenomena, for example, have both suffered from this problem during periods of their development. The physics of strong interactions is in such a period today. Naturally, our inability to extract useful predictions from a promising theory is a strong incentive to develop novel approximation techniques.

This article is devoted to a particular approximation scheme, commonly referred to as the method of "large-N expansions." The method is based on the possibly surprising fact that increasing the number of degrees of freedom can simplify the analysis of a theory. Specifically, if N is some measure of the number of degrees of freedom, or dimensionality, of a theory, then the theory may have a natural large-N generalization that we can solve explicitly in the limit $N \to \infty$. Such a solution allows one to calculate physical observables in the original finite-N theory by expanding them systematically in powers of $1/N$. This approach turns out to be applicable to a wide class of theories, including many for which conventional approximation techniques are inapplicable.

Large-N expansions were originally developed for specific theories in the areas of nuclear physics,[1] critical phenomena[2] and particle physics.[3] They subsequently found wide use in the analysis of a variety of theories in these fields[4] and in areas such as atomic physics[5] and quantum optics.[6]

Recent work has shown that all known examples of quantum theories in which large-N limits exist share a common underlying structure. In every case, one may regard the large-N limit as a new type of classical limit, that is, a limit in which the quantum dynamics become equivalent to classical dynamics. This means that one can find[7] a classical phase space and a classical Hamiltonian acting on this phase space, such that the resulting classical dynamics is equivalent to the dynamics of the original quantum theory in the limit $N \to \infty$. As a result, solving the quantum theory in the limit of infinite dimensionality is reduced to the problem of minimizing the equivalent classical Hamiltonian. I should emphasize that this large-N classical Hamiltonian is completely different from the conventional classical Hamiltonian that describes the $\hbar \to 0$ limit of a quantum theory. We will see specific examples that illustrate the difference.

In the discussion that follows, we will look at large-N expansions in greater detail. First we will illustrate the need for alternative approximation techniques by briefly examining several theories for which conventional techniques are inapplicable. Then we will discuss a number of examples of the application of large-N expansions to various types of quantum theories, and we will describe some of the explicit results that one may obtain. Finally, we will describe more fully the connection between large-N limits and classical dynamics, and illustrate this connection with the same set of examples.

Why bother with large N?

The most commonly used method for studying non-trivial quantum theories is undoubtedly perturbation theory. In many areas of physics, perturbative methods form the basis for almost all theoretical work. For example, all of the predictions of quantum electrodynamics—the most accurate physical theory known today—are derived from perturbative expansions in powers of the fine-structure constant α.

However, in certain types of quantum theories conventional perturbative techniques are completely useless. A particularly simple example is provided by a point particle moving in a quartic potential, $V(x) = \frac{1}{2}\alpha x^4$. If we start with the Schrödinger equation

$$[-(\hbar^2/2m)\nabla^2 + \tfrac{1}{2}\alpha x^4]\psi(\mathbf{x}) = E_i\,\psi(\mathbf{x})$$

and rescale the position, $\mathbf{x} \to (\hbar^2/m\alpha)^{1/6}\mathbf{x}$, it is easy to see that we can write the energy E_i of any eigenstate as

$$E_i = (\alpha\hbar^4/m^2)^{1/3}K_i$$

where K_i is a pure number, completely independent of the parameters α, $\hbar$ and m. In other words, any dimensionless ratio of energies, such as E_i/E_0, is a pure number that is not calculable by any sort of expansion in powers of the original parameters.

This same phenomenon occurs in quantum chromodynamics, which is widely believed to be the correct theory of the strong interactions. This theory describes the dynamics of quarks and gluons, and is expected to predict the binding of quarks into color singlet bound states. Superficially, QCD appears to contain a single coupling constant α_s, analogous to the coupling constant α in quantum electrodynamics. However, by a slightly more involved scaling transformation, one may show[8] that dimensionless ratios of physical quantities, such as ratios of the masses of different bound states, are pure numbers independent of the coupling constant α_s. Therefore, perturbative expansions in powers of α_s are useless for computing the masses or other physical properties of bound states.

A final example of this type of difficulty occurs in the theory of critical

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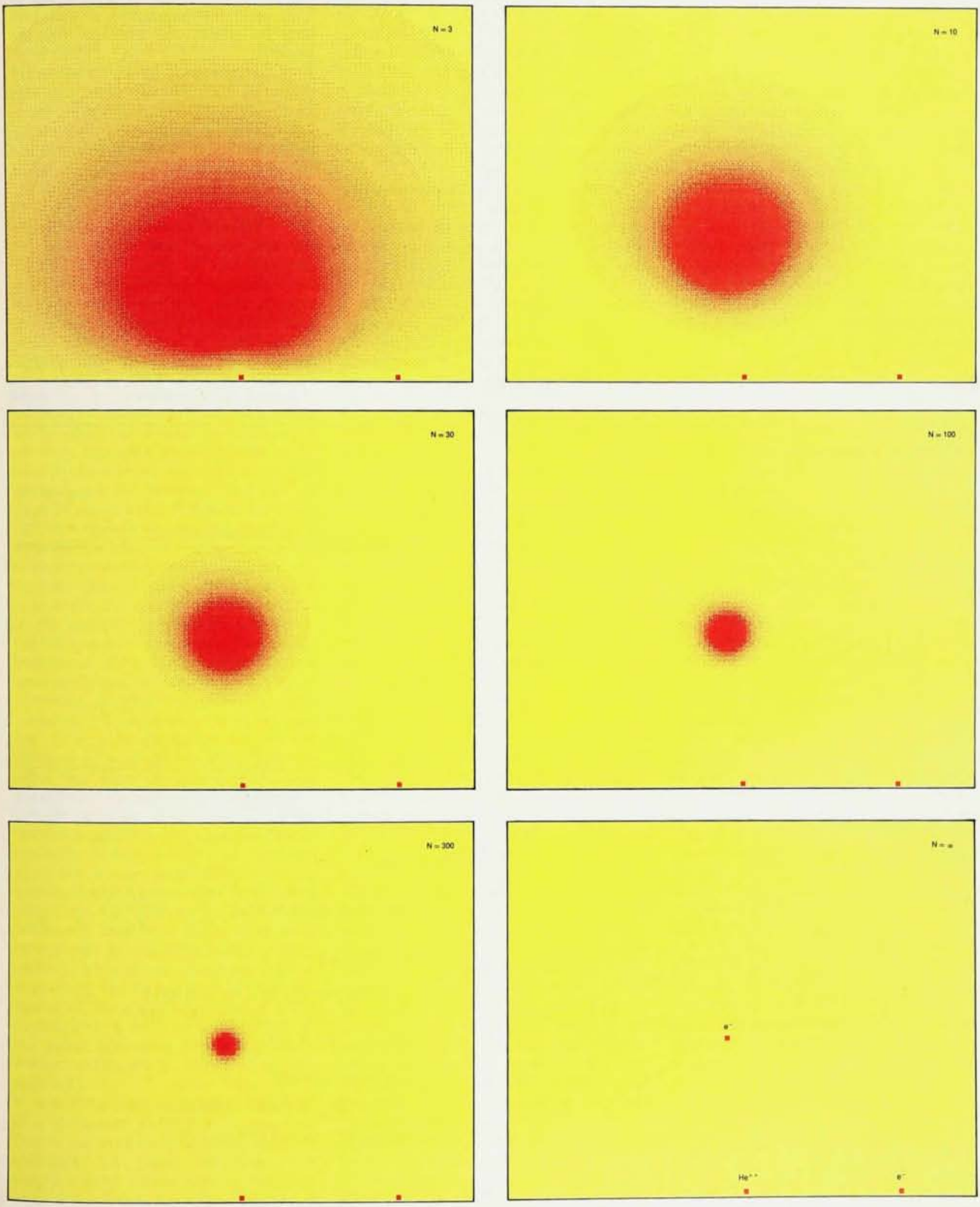


Probability distribution of an electron in a helium atom in N dimensions, for six different values of N. The density of points is proportional to the probability of finding one electron at a given position relative to the nucleus and the other electron. The two small squares at the bottom of each of these computer-drawn pictures represent the helium atom's nucleus (left) and the fixed electron (right). In each dimension, one electron is fixed at its root-mean-square distance from the nucleus. As the dimension increases, the probability distribution shrinks to a point. This is a consequence of the fact that the limit $N \to \infty$ is actually a novel type of classical limit. When N is infinite, one can compute the positions of the electrons directly by minimizing an appropriate classical Hamiltonian.

phenomena. Physicists have learned to describe the critical fluctuations that are present at typical second-order phase transitions by certain universal, scale-invariant field theories containing no parameters whatsoever. These theories characterize the behavior of the critical fluctuations with a set of critical exponents that, once again, are pure numbers incalculable by conventional perturbative techniques.

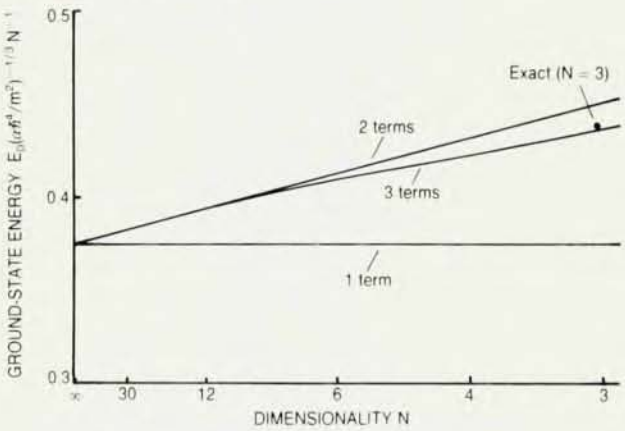
Now, in the case of a particle in a quartic potential, one can use the variational principle and directly construct wavefunctions that yield the ground-state energy, or other properties, to essentially arbitrary precision. Hence, one need not regard the lack of a systematic perturbative expansion as a serious obstacle to understanding the theory.

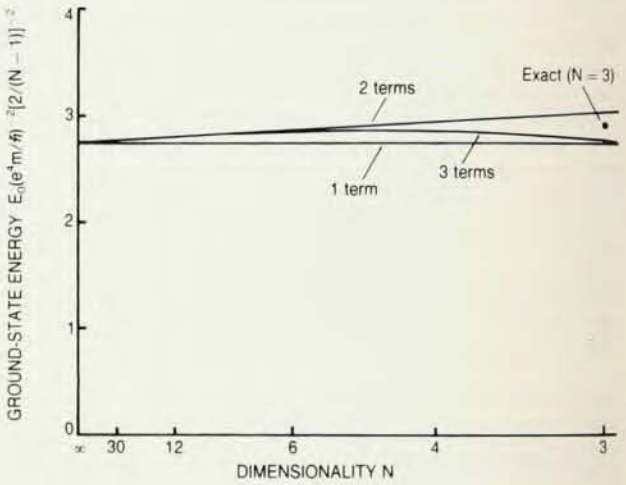
However, in most quantum field theories the inherent complexity of the theory makes the direct variational construction of an accurate approximation to the ground state virtually impossible. Consequently, in QCD, for example, the lack of any accurate, tractable approximation method has effectively blocked the calculation of low-energy phenomena. As a result, it is still not possible to confront with the real world what should be the theory's most basic prediction—the existence of hadronic bound states.

Large-N expansions represent an attempt to circumvent the failure of conventional perturbation theory by constructing a new expansion parameter "out of thin air." By generalizing a theory to a variable number of degrees of freedom one introduces a new parameter, $1/N$. If the generalized theory is soluble in the limit of infinite N, then a systematic expansion in powers of $1/N$ may be a very useful approach for studying the original finite-N theory and for calculating physical observables in that theory. Let us look at a few examples of quantum theories where one may use this approach:

► A particle moving in an arbitrary potential $V(x)$. We can generalize this naturally to a particle in N dimensions moving in a spherically symmetric potential $V(|\mathbf{x}|)$. Direct analysis of the radial Schrödinger equation allows us to compute the ground-state energy, or other properties, as a power series in $1/N$. Edward Witten, in his article (PHYSICS TODAY, July 1980, page 38) explains this in detail. In a similar way, multiparticle systems with rotationally invariant interactions may be generalized to N dimensions and solved in the limit $N \to \infty$. The box below shows some results for the quartic oscillator and the helium atom.

Particle in a potential





Quartic oscillator

Hamiltonian:

$$H = (\mathbf{p}\cdot\mathbf{p})/2m + (\alpha/2N)(\mathbf{x}\cdot\mathbf{x})^2$$

Ground-state energy:

$$E_0 = (\alpha\hbar^4/m^2)^{1/3}\{\tfrac{3}{8}N + (\sqrt{\tfrac{3}{2}} - 1) - (1/N)(\sqrt{\tfrac{2}{3}} - \tfrac{25}{36}) + \mathcal{O}(1/N^2)\}$$

$$= (\alpha\hbar^4/m^2)^{1/3}\{0.375N + 0.2247 - 0.1221(1/N) \times [1 + \mathcal{O}(1/N^3)]$$

$$\underset{N=3}{\to} (\alpha\hbar^4/m^2)^{1/3}[1.3173\ldots] \times \{0.8540 + 0.1706 - 0.0309 + \ldots\}$$

$$= 0.9937E_0^{\text{exact}}[1 + \mathcal{O}(1/N^3)]$$

Helium atom

Hamiltonian:

$$H = (\mathbf{p}_1\cdot\mathbf{p}_1)/2m + (\mathbf{p}_2\cdot\mathbf{p}_2)/2m - \frac{2e^2}{|\mathbf{r}_1|} - \frac{2e^2}{|\mathbf{r}_2|} + \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Ground-state energy:

$$E_0 = (e^4m/\hbar^2)\{2.7377[2/(N-1)]^2 + 0.2911 \times [2/(N-1)]^3 - 0.2744[2/(N-1)]^4\} \times (1 + \mathcal{O}[1/(N-1)]^3)$$

$$\underset{N=3}{\to} -(e^4m/\hbar^2)\,[2.9037\ldots] \times \{0.9428 + 0.1003 - 0.0945 + \ldots\}$$

$$= 0.9486E_0^{\text{exact}}(1 + \mathcal{O}[1/(N-1)]^3)$$

These results show the large-N behavior of the ground-state energy of the N-dimensional quartic oscillator and helium atom.[5] Each series is evaluated at $N = 3$ and expressed as the product of the known answer from numerical calculations and a series that should sum to unity. The figures show the partial sums of these series as functions of N. The helium-atom energy is expanded in powers of $1/(N-1)$ instead of $1/N$ because the ground-state energy is known to diverge as $N \to 1$. Therefore $1/(N-1)$ should be a more natural expansion parameter.

▶ Quantized scalar field theories. These so-called $\lambda\phi^4$ theories are often used to describe the interactions of spinless particles. One can generalize the theories to include scalar fields of N components $\phi_i(x)$ whose interactions are invariant under rotations in the N-dimensional space of fields. (The group of such rotations, denoted $O(N)$, is an example of a Lie group, or continuous group of transformations.) A quantized scalar field theory with $N = 3$ describes the critical behavior in, for example, isotropic (Heisenberg) ferromagnets; the theory with $N = 1$ describes uniaxial (Ising) ferromagnets. One can use standard functional integration techniques or other methods to solve these theories explicitly in the limit $N \to \infty$, and to compute corrections systematically in powers of $1/N$. Consequently, one can compute as a series in $1/N$ the critical exponents of N-component systems that are invariant under transformations in the group $O(N)$. This method for studying critical phenomena is the only approach known whose applicability is independent of the dimensionality of the system. The box on page 54 shows explicit results for some of the standard critical exponents.

▶ Quantum spin models. These are theories describing the dynamics of one or more quantum spins. Each spin of magnitude S has $(2S + 1)$ independent states, and the complete Hilbert space for the system is the product of all the individual $(2S + 1)$-dimensional state spaces. Acting on each spin is a set of operators obeying the standard commutation relations, and the Hamiltonian is a polynomial in all of the different spin operators. We can generalize this type of theory by simply considering the magnitude of the spins S to be variable. If we define N as $2S$, then once again we can solve the theory explicitly in the limit $N \to \infty$. We can apply an analogous treatment to spin models in which these spins form representations of arbitrary Lie groups. Models of this type have been used extensively in subfields ranging from nuclear physics to quantum optics. Results on the asymptotic large-N behavior of the energy spectra of these models give a good approximation to the exact spectrum, even for fairly modest values of N. (See the box on page 55.)

▶ A particle moving on the "manifold" of a group and subject to a potential that is an invariant function on that manifold. (A group manifold is a curved space whose points one can identify with the elements of some Lie group. A simple example is the manifold for the group SU(2), which is the surface of a sphere in four dimensions.) Theories based on group manifolds are also known as single-matrix models, because one can specify the position of a particle by a single matrix in the fundamental or defining representation of the group; in the Lie group SU(N), for example, this would be an $N \times N$ unitary matrix. One generalizes these theories by allowing the basic group manifold to vary over a sequence of groups such as O(N), SU(N) or Sp(N). Once again, observables such as the ground-state energy or excitation energies behave smoothly as $N \to \infty$, and one can compute the limiting asymptotics explicitly, as the box on page 57 shows. In the limit $N \to \infty$, these theories exhibit many features, including phase transitions, that occur in much more complicated theories, such as lattice gauge theories.

▶ Quantum chromodynamics. This is an SU(3) gauge theory containing quarks, which come in three types or "colors," and gluons, which come in eight types. We can generalize this theory naturally from an SU(3) theory to an SU(N) gauge theory, with N colors of quarks and $N^2 - 1$ gluons. The behavior of SU(N) gauge theories is believed to be very stable as $N \to \infty$. In fact, qualitative arguments suggest[10] that in the limit $N \to \infty$, the theory describes behavior surprisingly similar to that of the real world. For example, in the limit of large N, the theory predicts that there are infinitely many narrow resonances composed purely of valence quarks, that one-meson exchange dominates scattering amplitudes and that Zweig's rule is satisfied (this is the semiempirical rule that describes the strong suppression of hadronic processes that involve purely gluonic intermediate states). Despite considerable progress in understanding the theory, no one has yet solved explicitly the limit $N \to \infty$, and for this reason quantitative predictions are still lacking.

In each of these examples, the large-N limit simplifies the dynamics while simultaneously preserving the essential physical behavior of the original theory. Stable helium bound states, interesting ferromagnetic critical behavior, and a non-trivial meson spectrum, for example, all persist as $N \to \infty$. Such simplification is the primary utility of large-N expansions. Whether the $1/N$ expansion is quantitatively useful at a particular finite value of N naturally depends on the details of the theory. In the simpler examples of point particles or quantum spins (the boxes on pages 52 and 55), the first term alone frequently provides reasonable accuracy for physically interesting values of N. On the other hand, for critical phenomena the available large-N results (as shown in the box on page 54) are only qualitatively correct for real systems, for which $N \leq 3$. The accuracy of large-N results for quantum chromodynamics is totally unknown today.

$N \to \infty$ as a classical limit

The fact that in such a diverse class of theories one can increase the number of degrees of freedom in a way that simplifies the dynamics, naturally leads one to suspect some common underlying structure. This is indeed the case: We can understand every known large-N limit as a special type of classical limit. We can apply to large-N limits the same steps that are used to derive classical mechanics as the $\hbar \to 0$ limit of quantum mechanics.

It is appropriate, then, to discuss briefly the quantum mechanics of a point particle in the limit $\hbar \to 0$. Then we can generalize the key steps in this treatment in a way that produces a completely general prescription for constructing classical limits in arbitrary quantum theories. Finally, we can apply this prescription to various large-N theories to construct explicitly the classical dynamics that reproduces the $N \to \infty$ limit of the original quantum theory.

A classical limit in any quantum theory is a limit in which all quantum interference effects disappear. To demonstrate this in any particular theory one must be able to construct a special basis of states $|z\rangle$, which might be called "classical" or "coherent" states, and show that any coherent superposition of these states, such as $\alpha|z\rangle + \alpha'|z'\rangle$ becomes in the limit physically indistinguishable from an incoherent mixture of these states, represented by a density matrix $|\alpha|^2|z\rangle\langle z| + |\alpha'|^2|z'\rangle\langle z'|$. Furthermore, one should then be able to identify each of these special states $|z\rangle$ with some point in a classical phase space, and for some choice of classical Hamiltonian show that the original quantum dynamics becomes equivalent to classical dynamics on this phase space.

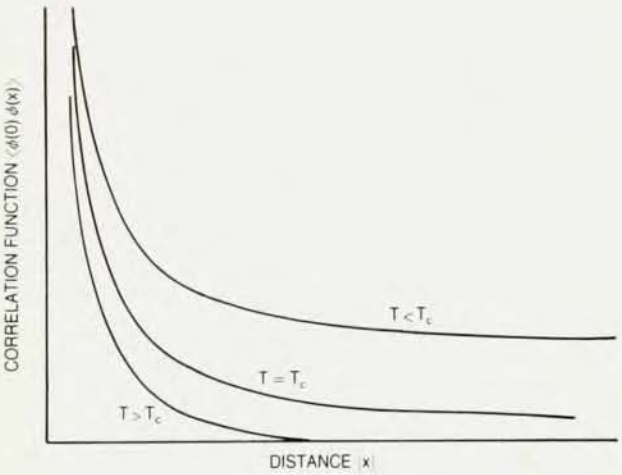
Naturally, we may use this procedure to demonstrate that the usual $\hbar \to 0$ limit for a point particle in an arbitrary potential is a classical limit. Clearly, we should choose the "classical" states to describe wave packets that become increasingly localized in both position and momentum as $\hbar \to 0$. It is particularly convenient to choose the well-known Gaussian coherent states $|\,|p,q\rangle\,|$, with wavefunctions given by

$$\langle x|\,p,q\rangle = (\pi\hbar)^{-1/4}\exp\{(1/\hbar)[ipx - \tfrac{1}{2}(x-q)^2]\}$$

Note that the labels p and q simply specify the mean momentum and position of the coherent state:

$$\langle p,q|\,\hat{p}\,|p,q\rangle = p$$
$$\langle p,q|\,\hat{x}\,|p,q\rangle = q$$

Critical exponents



N-component critical phenomena

Hamiltonian:

$$H \equiv \int \mathrm{d}^{d-1}x \, \tfrac{1}{2}\{[\pi(x)]^2 + [\nabla\phi(x)]^2 + \mu^2[\phi(x)]^2 + (\lambda/2N)[\phi(x)\cdot\phi(x)]^2\}$$

Correlation function:

$$G(x) \equiv (1/N)\langle\phi(0)\cdot\phi(x)\rangle$$

$$\lim_{N\to\infty} G(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{ik\cdot x}}{(k^2+m^2)}$$

Gap equation:

$$m^2 = \mu^2 + \lambda G(0)$$

$$T - T_c \propto m^2$$

Physical observable	Definition	Critical behavior	Exponent (d = 3)
Critical correlation	$G(x)$	$G(x) \sim \lvert x\rvert^{-(d-2+\eta)}$, $T = T_c$	$\eta = \frac{1}{3}[8/(\pi^2 N)] - (\frac{2}{3})^2[8/(\pi^2 N)]^2 + \mathcal{O}(1/N^3)$
Correlation length	$\xi \equiv \lim_{x\to\infty}(-G(x)/x)^{-1}$	$\xi \sim (T - T_c)^{-\nu}$, $T \gtrsim T_c$	$\nu = 1 - \frac{4}{3}[8/(\pi^2 N)] + \mathcal{O}(1/N^2)$
Susceptibility	$\chi \equiv \int \mathrm{d}^d x\, G(x)$	$\chi \sim (T - T_c)^{-\gamma}$, $T \gtrsim T_c$	$\gamma = 2 - 3[8/(\pi^2 N)] + \mathcal{O}(1/N^2)$

These are large-N expansions of selected critical exponents in three dimensions. Critical exponents characterize the behavior of the correlation function $G(x)$ near the critical point T_c. The figure shows typical behavior of the correlation function. It is easy to extract the limiting values of the exponents from the $N\to\infty$ limit of $G(x)$, given above. The gap equation implicitly determines the physical mass m. The critical point is defined as the point where m^2 vanishes. One may systematically compute corrections to the limiting values of critical exponents.[4] For $N = 3$, the first two terms of these series yield critical exponents $\nu = 0.64$, $\gamma = 1.2$ and $\eta = 0.068$, whereas the experimental values are approximately $\nu = 0.71$, $\gamma = 1.4$ and $\eta = 0.04$.

This set of states provides a complete basis, so that any state may be expressed as a linear combination of coherent states. The basis is "overcomplete" in that there are more than a countable number of these states, and states for different values of p and q are not orthogonal; in other words, their wave functions overlap to some extent.

The overlap of any two different coherent states is Gaussian in form and decreases exponentially as $\hbar \to 0$. Furthermore, matrix elements between different coherent states of any reasonable operator $\hat{A}$ vanish as $\hbar \to 0$, a reasonable operator being one such as any polynomial in $\hat{x}$ and $\hat{p}$ with no explicit $\hbar$ dependence. In other words, $\langle p,q|\hat{A}|p',q'\rangle / \langle p,q|p',q'\rangle$ has a finite limit as $\hbar \to 0$. Consequently, measurements with any reasonable operator cannot distinguish a superposition of different coherent states from a mixed state as $\hbar \to 0$, and quantum interference effects between different coherent states vanish as $\hbar \to 0$.

At this point, one may associate each coherent state $|p,q\rangle$ with the point (p,q) in the usual two dimensional classical phase space, so that expectations of quantum operators $\hat{A}$ become ordinary functions a on the classical phase space:

$$a(p,q) = \lim_{\hbar\to 0} \langle p,q|\hat{A}|p,q\rangle$$

Finally, one can show that expectations of products of quantum operators factorize in the limit

$$\lim_{\hbar\to 0} \langle p,q|\hat{A}\hat{B}|p,q\rangle = a(p,q)b(p,q)$$

and that commutators of quantum operators, divided by $\hbar$, become classical Poisson brackets, which we will denote by curly brackets:

$$\lim_{\hbar\to 0} (i/\hbar)\langle p,q|[\hat{A}\hat{B}]|p,q\rangle = \frac{\partial a(p,q)}{\partial p}\frac{\partial b(p,q)}{\partial q} - \frac{\partial a(p,q)}{\partial q}\frac{\partial b(p,q)}{\partial p} \equiv \{a(p,q), b(p,q)\}$$

This implies that the quantum equations of motion, $\partial\hat{A}/\partial t = (i/\hbar)[\hat{H},\hat{A}]$, reduce to the classical Hamilton equations, $\partial a(p,q)/\partial t = \{h(p,q), a(p,q)\}$. In this fashion, one can derive conventional classical mechanics as the $\hbar \to 0$ limit of the original quantum theory.

One may use essentially the same approach to construct other classical limits in arbitrary quantum theories. In each case one must construct an appropriate set of generalized coherent states and show that these states behave classically in the limit. To learn how to construct useful generalizations of the standard Gaussian coherent states, it is important to observe that these coherent states may be defined by the action of the Heisenberg group, that is, the group of unitary transformations

$$U(p,q,\alpha) \equiv \exp(i\alpha/\hbar)\exp(ip\hat{x}/\hbar) \times \exp(-iq\hat{p}/\hbar)$$

applied to a single basic state $|0\rangle$. The parameter α is a phase factor that is required so that this set of operators forms a group.

If one chooses the basic state to have a simple Gaussian wavefunction

$$\langle x|0\rangle = (\pi\hbar)^{-1/4}\exp -(x^2/2\hbar)$$

then elements of the Heisenberg group precisely generate the previous coherent states up to an overall phase factor

$$\hat{U}(p,q,\alpha)|0\rangle = e^{i\alpha/\hbar}|p,q\rangle$$

One may regard the fact that the Gaussian coherent states form a complete basis to be a consequence of the fact that the Heisenberg group acts

irreducibly on the Hilbert space of the particle. This means that there is no nontrivial subspace of the Hilbert space that is left invariant under the action of all elements of the Heisenberg group, or equivalently, that the only operators that commute with all elements of the group are proportional to the identity. This irreducibility immediately implies the existence of the coherent-state completeness relation

$$\hat{1} = (1/2\pi\hbar)\int dpdq|\, p,q\rangle\langle p,q|$$

A general prescription. The observations in the preceeding two paragraphs motivate the following general prescription for constructing classical limits in arbitrary quantum theories.[7]

▶ Find a Lie group G of unitary operators that acts irreducibly on the Hilbert space of the theory. We will call this group G the coherence group. We can represent elements of the coherence group as $|\exp it_\alpha \hat{\Lambda}_\alpha|$, where the set of operators $|\hat{\Lambda}_\alpha|$ forms a basis for the Lie algebra of the group G. In other words, they satisfy the commutation relations $i[\hat{\Lambda}_\alpha,\hat{\Lambda}_\beta] = c_{\alpha\beta}{}^\gamma \hat{\Lambda}_\gamma$ for some set of "structure constants" $|c_{\alpha\beta}{}^\gamma|$ that reflect the geometry of the Lie group. For the Heisenberg group, this basis is simply the set of operators $\hat{x}/\hbar$, $\hat{p}/\hbar$ and $1/\hbar$.)

▶ Construct a set of coherent states $|\,|u\rangle|$ by applying each of the elements of the coherence group to some initial state $|0\rangle$: $|u\rangle \equiv \hat{U}|0\rangle$ for each element $\hat{U}$ in G. The irreducibility of the group G automatically implies that this set of coherent states forms a complete basis.

▶ Show that in some limit of the theory, different coherent states become orthogonal. Specifically, if a parameter χ of the theory such as $\hbar$ or $1/N$ vanishes in the limit, then the overlap of different coherent states must decrease exponentially as χ tends to zero.

These three steps are actually sufficient to demonstrate that the limit $\chi \to 0$ is a classical limit. If one defines "classical" or "reasonable" operators to be operators $\hat{A}$ whose coherent-state matrix elements $\langle u|\hat{A}|u'\rangle/\langle u|u'\rangle$ have finite limits as $\chi \to 0$, then one may show that the operators $|\chi\hat{\Lambda}_\alpha|$, or arbitrary polynomials in these operators with no explicit χ dependence, are classical operators. Measurements with the set of operators $|\chi\hat{\Lambda}_\alpha|$ distinguish different coherent states. However, the fact that different coherent states become orthogonal as $\chi \to 0$ implies that no measurement with any classical operator can distinguish a superposition of different coherent states from a mixed state as $\chi \to 0$. This shows that quantum interference effects between different coherent states vanish as $\chi \to 0$.

One now associates each coherent state with some point on a classical phase space. A set of coordinates $|\xi_\alpha|$ on the phase space is provided by the expectations of the operators $|\chi\hat{\Lambda}_\alpha|$; in other words

$$\xi_\alpha \equiv \lim_{\chi\to 0}\langle u|\chi\hat{\Lambda}_\alpha|u\rangle$$

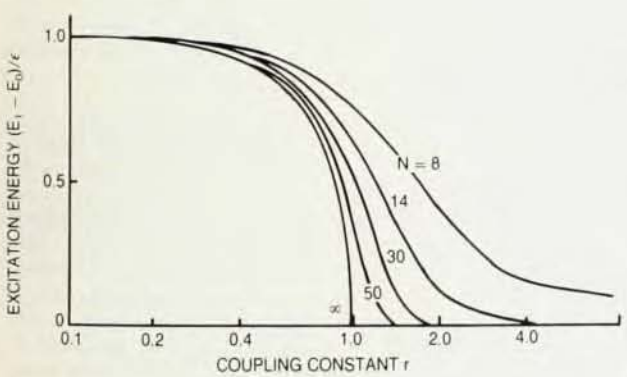
defines the coordinates of the point associated with the coherent state $|u\rangle$. (One may also describe the classical phase space in a geometrical, coordinate-free manner. Such a description is given by a subspace known as a "coadjoint orbit" of the Lie algebra of G.) Because the coordinates $|\xi_\alpha|$ distinguish the various coherent states, the expectation of any reasonable operator $\hat{A}$ defines an ordinary function on the classical phase space,

$$\lim_{\chi\to 0}\langle u|\hat{A}|u\rangle \equiv a(\xi)$$

Once again, the absence of quantum interference as $\chi \to 0$ implies that expectations of products of operators factorize

$$\lim_{\chi\to 0}\langle u|\hat{A}\hat{B}|u\rangle = a(\xi)b(\xi)$$

Quantum spins



Large-N quantum spins

Hamiltonian:

$$H = \epsilon[\hat{S}_z + (r/N)(\hat{S}_x{}^2 - \hat{S}_y{}^2)],\quad \hat{S}^2 = (N/2)(N/2+1)$$

Excitation energy:

$$\lim_{N\to\infty}(E_1 - E_0)/\epsilon = \begin{cases}(1-r^2)^{1/2} & |r|<1\\ 0 & |r|>1\end{cases}$$

Ground-state energy:

$$\lim_{N\to\infty} E_0/(\tfrac{1}{2}\epsilon N) = \min_{|\theta,\phi|}[\cos\theta + \tfrac{1}{2}r\sin^2\theta\cos 2\phi] = \begin{cases}-1 & |r|<1\\ -\tfrac{1}{2}|r+1/r| & |r|>1\end{cases}$$

Values of ground-state energy $-E_0/(\frac{1}{2}\epsilon N)$

	$N=\infty$	$N=30$	$N=8$	$N=2$
$r=0.0$	1.00	1.000	1.000	1.000
$r=0.4$	1.00	1.003	1.009	1.020
$r=1.0$	1.00	1.021	1.058	1.118
$r=2.0$	1.25	1.236	1.258	1.414
$r=5.0$	2.60	2.537	2.390	2.693
$r\to\infty$	$0.5\,r$	$0.487\,r$	$0.451\,r$	$0.5\,r$

These results show the large-N behavior for a theory of a single quantum spin. This theory is equivalent to the Lipkin–Meshkov–Glick Hamiltonian,

$$H = \tfrac{1}{2}\epsilon\left(\sum_{p,\sigma}\sigma a^\dagger_{p,\sigma}a_{p,\sigma} + (r/N)\sum_{p,p',\sigma}a^\dagger_{p,\sigma}a^\dagger_{p',\sigma}a_{p',-\sigma}a_{p,-\sigma}\right)$$

which models the interactions of N nucleons distributed among two different N-fold degenerate levels. The index σ takes on values of $+1$ and -1, corresponding to upper and lower levels, respectively; the integer p runs from 1 to N. Shown is the behavior of the ground-state energy and the excitation energy to the first excited state as a function of the number N of nucleons and the coupling constant r. Note how the variation with coupling constant becomes increasingly rapid as N increases, and how in the limit $N\to\infty$ it produces nonanalyticity at $r=1$. (Numerical data and figure adapted from reference 1.)

Furthermore, one may study the behavior of commutators of quantum operators, divided by γ, and show that

$$\lim_{\gamma\to 0}\frac{i}{\gamma}\langle u|[\hat{A},\hat{B}]|u\rangle = \frac{\partial a(\xi)}{\partial\xi_\alpha}\frac{\partial b(\xi)}{\partial\xi_\beta}c_{\alpha\beta}{}^{\gamma}\xi_\gamma \equiv |a(\xi),b(\xi)| \quad (1)$$

Here we assume summation over the repeated indices. The right-hand side of this equation can be defined to be the Poisson bracket of the classical phase space. Remarkably, all of the structure needed to define the classical phase space follows directly from the structure of the coherence group.

Note that equation 1 implies that the quantum equations of motion, $\partial\hat{A}/\partial t = i[\hat{H},\hat{A}]$, reduce to the classical Hamilton equations, $\partial a(\xi)/\partial t = |h_{\rm cl}(\xi),a(\xi)|$ with the classical Hamiltonian given by

$$h_{\rm cl}(\xi)\equiv\lim_{\gamma\to 0}\gamma\langle u|\hat{H}|u\rangle \quad (2)$$

provided $\gamma\hat{H}$ is a "reasonable" quantum operator.

This shows that if we can carry out for a quantum theory the three steps of our general prescription, then that theory has a classical limit. From the classical dynamics based on the Hamiltonian in equation 2, we can extract the behavior of any physical observable as $\gamma\to 0$. For example, to calculate the low-energy spectrum of the theory, we note that the ground-state energy is given by

$$E_0 \simeq (1/\gamma)\epsilon_0$$

where ϵ_0 is the minimum of the classical Hamiltonian $h_{\rm cl}(\xi)$. Expanding the classical equations of motion, $\dot{\xi} = |h_{\rm cl},\xi|$, about this minimum yields a set of small-oscillation frequencies $|\omega_i|$. These frequencies directly determine the $\gamma\to 0$ limit of the low-energy spectrum of the theory; the excitation energy to any low-lying excited state is given by $\Delta E = \Sigma n_i\omega_i$ for some set of nonnegative integers $|n_i|$.

Some simple examples

Let us now see how our prescription for constructing classical limits applies to various large-N theories. A single particle moving in an N-dimensional central potential provides the simplest example. Because the Hamiltonian is spherically symmetric, we may completely limit consideration to the rotationally invariant, or zero angular momentum sector of the theory. (One can carry out a completely analogous discussion in any fixed-angular-momentum subspace of the theory.) We will require all physical operators to be rotationally invariant. To construct a useful set of coherent states, we must first find a suitable coherence group that acts irreducibly. The simplest choice is the group generated by the operators $\mathbf{x}\cdot\mathbf{x}$ and $(\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x})$; our group G is thus the set of operators

$$\hat{U}(a,b)\equiv\exp(i/2)[a\mathbf{x}\cdot\mathbf{x}+b(\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x})]$$

No rotationally invariant operator, except constants, commutes with all elements of this group, and consequently it acts irreducibly. To construct coherent states one applies elements of this group to some initial state $|0\rangle$. A particularly simple choice for the initial state is the standard Gaussian, given by

$$\langle\mathbf{x}|0\rangle = C\exp(-\tfrac{1}{2}\mathbf{x}\cdot\mathbf{x})$$

For this choice, it is straightforward to evaluate the resulting coherent-state wavefunctions $|u\rangle\equiv\hat{U}(a,b)|0\rangle$. These are simply new Gaussians

$$\langle\mathbf{x}|u\rangle = C(z)\exp(-\tfrac{1}{2}z\mathbf{x}\cdot\mathbf{x})$$

where $z = e^{2b} - ia(e^{2b}-1)/2b$, and $C(z)$ is a normalization constant. The overlap of any two of these states is

$$\langle u_1|u_2\rangle = [(z_1^*+z_2)^2/(z_1^*+z_1)(z_2^*+z_2)]^{-N/4}$$

which decreases exponentially as $N\to\infty$ if $z_1\neq z_2$. Thus, if the parameter γ in the general discussion is chosen to be $1/N$, then all the requirements are satisfied for the limit $N\to\infty$ to be a classical limit.

If we define w as $1/(2\mathrm{Re}\,z)$ and v as $-\mathrm{Im}\,z$, then the generators of the coherence group are given by

$$(1/N)\langle u|\mathbf{x}\cdot\mathbf{x}|u\rangle = w \text{ and}$$

$$(1/N)\langle u|\mathbf{x}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{x}|u\rangle = 2vw$$

The parameters v and w may be used as coordinates on the classical phase space. The definition of the Poisson bracket in equation 1 implies that v and w are canonically conjugate: $|v,w| = 2$. If the quantum Hamiltonian is given by $\hat{H} = \frac{1}{2}\mathbf{p}\cdot\mathbf{p} + NV(\mathbf{x}\cdot\mathbf{x}/N)$, then the classical Hamiltonian is

$$h_{\rm cl}(v,w)\equiv\lim_{N\to\infty}\langle u|\hat{H}|u\rangle = \tfrac{1}{2}v^2w + \tfrac{1}{8}w^{-1} + V(w)$$

For large N, the ground-state energy E_0 is given by $N\epsilon_0$, where ϵ_0 is $h_{\rm cl}(v_0,w_0)$, the minimum of the classical Hamiltonian. Small fluctuations about the minimum are described by the differential equation $\ddot{\delta} = -\omega^2\delta$, where δ is $w-w_0$, and ω^2 is proportional to the curvature of the classical Hamiltonian at the minimum. The frequency ω is the $N\to\infty$ limit of the excitation energy to the first rotationally invariant excited state.

This calculation of the low-energy spectrum for a particle in an N-dimensional central potential illustrates the use of coherent-state methods to demonstrate the classical nature of the large-N limit. This particular example is sufficiently simple that we could have calculated the large-N behavior of, for example, the ground-state energy equally easily by directly examining the Schrödinger equation written in radial coordinates. The most important virtue of the coherent-state method is, however, the ease with which we may apply it to more complex problems. In every case it is easy to carry out the basic procedure of constructing a suitable set of coherent states and checking that they become orthogonal as $N\to\infty$. "Solving" the theory for infinite N is then reduced to the straightforward task of minimizing the resulting classical Hamiltonian for $N=\infty$.

Helium atom. As an example, let us look at the derivation of the large-N behavior of a helium atom, that is, two particles moving in a Coulomb potential in N dimensions. Here one simply replaces the previous coherence group by the group generated by the set of operators $(\mathbf{x}_i\cdot\mathbf{x}_j)$ and $(\mathbf{x}_i\cdot\mathbf{p}_j+\mathbf{p}_j\cdot\mathbf{x}_i)$, where i and j can be 1 or 2. Acting on the standard Gaussian wavefunction $\exp-\frac{1}{2}(\mathbf{x}_1{}^2+\mathbf{x}_2{}^2)$ this group generates a simple set of Gaussian coherent states

$$\langle\mathbf{x}_1,\mathbf{x}_2|u\rangle = C(z)\exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{2}\mathbf{x}_i\cdot z_{ij}\mathbf{x}_j\right]$$

where z is now a 2×2 complex symmetric matrix. Once again, these states become orthogonal as $N\to\infty$. The classical Hamiltonian that follows from the quantum helium Hamiltonian

$$\hat{H} = \mathbf{p}_1{}^2/2m + \mathbf{p}_2{}^2/2m - N^{3/2}(2e^2/|\mathbf{r}_1| + 2e^2/|\mathbf{r}_2| - e^2/|\mathbf{r}_1-\mathbf{r}_2|)$$

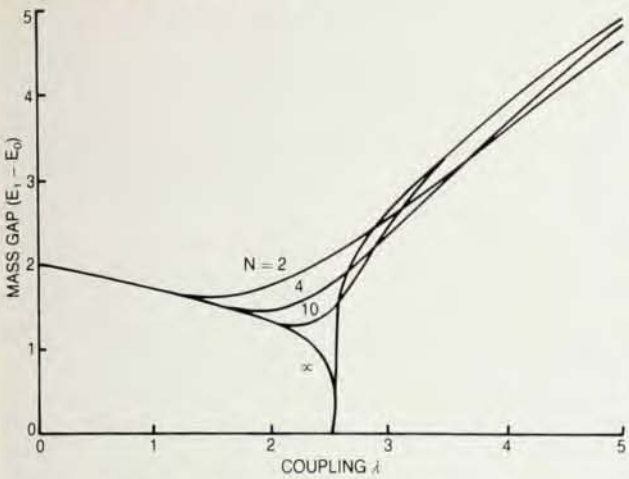
is found to be

$$h_{\rm cl} = (\hbar^2/2m)\,\mathrm{tr}[vwv + \tfrac{1}{4}w^{-1}] - e^2[2(w_{11})^{-1/2} + 2(w_{22})^{-1/2} - (w_{11}+w_{22}-2w_{12})^{-1/2}]$$

The parameters w and v, defined above in terms of Re z and Im z, respectively, are now 2×2 real symmetric matrices, and are canonically conjugate coordinates on the classical phase space. Minimizing this Hamiltonian leads to the first term in the large-N expansion of the ground-state energy given in the box on page 52.

Quantum-spins. The natural coherence group for quantum spin models is the group generated by all linear combinations of the fundamental spin operators. In other words, the coherence group consists of independent rotations of each quantum spin. Acting on an initial state where all spins are in the highest state possible (in general, a "maximal weight vector"), this group generates a set of coherent states in which each spin points in an arbitrary direction. As $N\to\infty$ various coherent states become orthogonal and the quantum spin model reduces to a system of

Particle on a group manifold



U(*N*) matrix model

Hamiltonian:

$$H \equiv \mathrm{tr}[(\lambda/N)\mathbf{E}^2 + (N/\lambda)(2 - \mathbf{V} - \mathbf{V}^\dagger)],\quad \mathbf{V}\in \mathrm{U}(N),\quad \lambda_c = 8/\pi$$

Ground-state energy:

$$\lim_{N\to\infty} E_0/N^2 = -\frac{\lambda}{12} + (1/\lambda)(2/3k)^2\left[(1+4k^2) - 2(\lambda/\lambda_c)(1-1/k^2)\begin{cases} k\,K(k), & \lambda > \lambda_c \\ K(1/k), & \lambda < \lambda_c \end{cases}\right]$$

Mass gap:

$$\lim_{N\to\infty} E_1 - E_0 = \begin{cases} 2\pi[kK(k)]^{-1}, & \lambda > \lambda_c \\ \pi[K(1/k)]^{-1}, & \lambda < \lambda_c \end{cases}$$

$$\frac{\lambda}{\lambda_c} \equiv \begin{cases} E(k)/k, & \lambda > \lambda_c \\ E(1/k) - (1-1/k^2)K(1/k), & \lambda < \lambda_c \end{cases}$$

The graph and equations above give some characteristics of a particle moving on the manifold of the group U(N), for large values of the dimensionality N. In the Hamiltonian, $\mathbf{V}$ is an $N\times N$ unitary matrix that represents the coordinates of a particle on the manifold of the group U(N), and $\mathbf{E}$ is an $N\times N$ matrix of conjugate momenta. The other equations give the ground-state energy E_0/N^2 and the mass gap $(E_1 - E_0)$, in the limit $N\to\infty$. The functions $E(k)$ and $K(k)$ are complete elliptic integrals; the last equation shown determines the parameter k implicitly. The U(N) matrix model is exceptional in that one may solve it exactly[9] for finite N. The ground state energy (divided by N^2) is an extremely smooth function of N. In fact for all couplings λ and all $N\geq 2$, the ground-state energy never differs from its $N=\infty$ limit by more than 2%. The graph shows the mass gap $(E_1 - E_0)$ as a function of coupling λ for $N = 2$, 4, 10 and ∞. As in the quantum spin model (the box on page 55), the crossover from weak to strong coupling behavior becomes increasingly rapid as N increases, and, for $N=\infty$, becomes a genuine phase transition at $\lambda = \lambda_c$.

classical spins. For a single SU(2) quantum spin, the classical phase space is simply a two-dimensional sphere, and the quantum spin operators $\hat{\mathbf{S}}$ become $N/2$ times a three-component unit vector $\mathbf{s}$, which describes the classical spin:

$$\langle u|\hat{\mathbf{S}}|u\rangle \simeq (N/2)\mathbf{s}$$

This leads easily to the results for large N that are shown on page 55.

Particle on a group manifold. The only essential difference between gauge theories or matrix models and the previous examples is in the complexity of their coherence groups and classical phase spaces. Even for a single-matrix model, such as a particle moving on the group manifold of U(N) (or equivalently, a gauge theory defined on a lattice containing four points at the corners of a square), the coherence group is necessarily of infinite dimension. If the basic operators are the matrix $\hat{\mathbf{V}}$ and its conjugate momentum $\hat{\mathbf{E}}$, then one may generate a suitable coherence group by the set of operators $\mathrm{tr}(\hat{\mathbf{V}}^k)$ and $\mathrm{tr}(\hat{\mathbf{E}}\hat{\mathbf{V}}^k + \hat{\mathbf{V}}^k\hat{\mathbf{E}})$ for all integer powers k. One can explicitly construct the coherent states that this group generates when it acts on a simple initial state defined by $\hat{\mathbf{E}}|0\rangle = 0$, and thereby derive the infinite-dimensional classical phase space. Convenient coordinates are provided by a function $\omega(\theta)$, which is the density of eigenvalues of the matrix $\hat{\mathbf{V}}$, and its conjugate momentum $v(\theta)$. The classical Hamiltonian corresponding to a quantum Hamiltonian such as the one given in the box above is

$$h_{\mathrm{cl}} = \int_{-\pi}^{\pi} d\theta\, \omega(\theta)[\lambda v^2(\theta) + (\pi^2/3)\lambda\omega(\theta)^2 - (2/\lambda)\cos\theta - \lambda/12]$$

Minimizing this Hamiltonian subject to the constraint $\int d\theta\omega(\theta) = 1$ leads to the results shown in the box.

Finally, for gauge theories, the requirement that the coherence group act irreducibly leads one to the group generated by the set of operators $\mathrm{tr}(\hat{\mathbf{V}}_{xx}^k)$ and $\mathrm{tr}[\hat{\mathbf{E}}(x)\hat{\mathbf{V}}_{xx}^k + \hat{\mathbf{V}}_{xx}^k\hat{\mathbf{E}}(x)]$. Here $\hat{\mathbf{V}}_{xy}$, which is defined as

$$\exp\int_{\Gamma_{xy}} \hat{A}\cdot dx$$

is the exponential of the integral of the gauge field $\hat{A}$ along on an arbitrary path Γ_{xy} connecting the points x and y, and the generators of the coherence group include all possible closed loops Γ_{xx}. Needless to say, this group generates a very complicated set of coherent states. Nevertheless, one can show that the limit $N\to\infty$ is a classical limit for these states.[7]

Unfortunately, it has not yet been possible to minimize explicitly the resulting classical Hamiltonian. Therefore, despite the fact that the $N\to\infty$ quantum theory has been reduced to a classical theory, we still lack explicit quantitative predictions. This is an area of active research today, and we will likely see further progress soon.

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