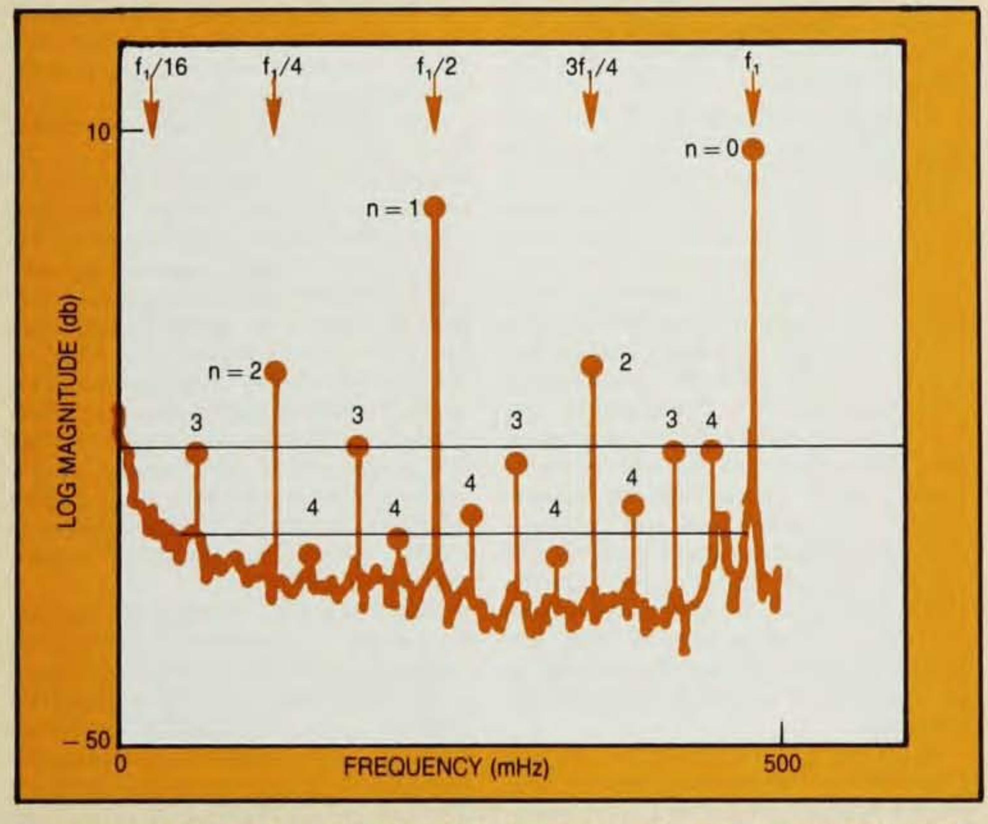
Period-doubling route to chaos shows universality



Experimental observation of period-doubling behavior leading to turbulence in Rayleigh–Benard flow. Libchaber and Maurer found frequencies f_1, $f_1/2$, $f_1/4$, $f_1/8$ and $f_1/16$. The set of subharmonic spectral peaks that have appeared at the nth period doubling are so labeled. Using the $n = 2$ peaks the theoretically predicted averages for the $n = 3$ and 4 peaks are drawn as horizontal lines. The two lines are 8.2 db apart. Figure adapted from references 3 and 4.

In some systems, as an external parameter is varied, the behavior of the system changes from simple to erratic. For some range of the parameter, the behavior of the system is orderly; for example, it is periodic in time with a period, T. Beyond this range, the behavior does not reproduce itself in T seconds. Instead, two intervals of T are required. In order words, the period doubles to $2T$. This new periodicity remains over some range of parameter values until another critical value is reached; now a period of $4T$ is required for reproduction. At a certain value of the parameter, an infinite number of doublings has been reached and the behavior has become aperiodic or chaotic. Thus period doubling is a characteristic route for certain systems to follow as they change from simple periodic to complex aperiodic motion.

Surprisingly, this mathematical curiosity of period doubling turns out to occur in many real physical systems. Among the systems now believed to show period-doubling behavior are some biological populations and a large variety of noisy mechanical, electrical and chemical oscillators, which are examples of coupled sets of nonlinear differential equations. An experiment on Rayleigh–Benard flow shows evidence for periods geometrically increasing up to a factor of 16. Very recently suggestions of period-doubling behavior have been reported for noisy Josephson junctions and for optically bistable cavities. All these period-doubling systems are believed to have universal behavior, analogous to the well-developed theory of critical phenomena.

History. In 1973 Nicholas Metropolis, Myron Stein and Paul Stein (Los Alamos) discovered a qualitative universal behavior in period-doubling one-dimensional maps or iterative schemes. In connection with population biology studies, Robert May (Princeton) and George Oster (Berkeley) noted in 1974 that period doubling can lead to chaos. The following year Mitchell Feigenbaum (Los Alamos) became interested in period doubling after hearing a talk by Steven Smale (Berkeley), who considered a connection between an infinite number of period doublings and a strange attractor. A strange attractor is a region of phase space that a system point's trajectory must enter. On its very complicated, multiply connected spongelike surface, nearby trajectories diverge from each other. David Ruelle (Institut des Hautes Etudes Scientifiques) and Floris Takens had suggested that these strange attractors might describe chaos.

Feigenbaum started studying one-dimensional maps, changing x_j to x_{j+1} by a quadratic transformation

$$x_{j+1} = \lambda x_j(1 - x_j)$$

with a fixed value of the external parameter λ. Using a programmable pocket calculator, Feigenbaum numerically determined some parameter values at which period doubling occurs and immediately saw that the values of λ_n were converging geometrically. (For λ greater than λ_n but less than λ_{n+1}, there is a stable period of frequency 2^n.) Feigenbaum believes he was the first to find this geometric convergence because previous calculations of λ_n had always been performed automatically on large, fast computers. He found the convergence rate was 4.669. Then, instead of a quadratic transformation, he tried a trigonometric transformation, and to his surprise the λ_n again converged geometrically at the same rate—4.669. Thus, he concluded that for one-dimensional transformations, as you change the parameter λ, if you take the ratio of two successive parameter changes, the ratio approaches a universal value, δ. That is,

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}} \xrightarrow[n\to\infty]{} \delta = 4.6692016\ldots$$

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in the limit as the number of doublings approaches infinity.

Feigenbaum soon found another universal property. Each time the period doubles, some features take twice as many steps, $2n$. So if you only went n steps, you would have an error. The ratio of successive errors in the limit of large n (in other words, a fundamental rescaling) is a universal constant, 2.502907875.... Using these ingredients, Feigenbaum developed a detailed theory of these universal phenomena that includes a description of the dynamics. He reported these results at a number of meetings in 1976 and 1977 and then in the *Journal of Statistical Physics* in 1978.[1]

Feigenbaum told us the value for δ appears as a natural rate in all systems exhibiting a period-doubling route to chaos. "In fact," he says, "most measurable properties of any such system in this aperiodic limit now can be determined in a way that essentially bypasses the details of the equations governing each specific system because the theory of this behavior is universal over such details." Provided the system has the qualitative properties allowing it to take this route to chaos, its quantitative properties are determined. He notes that the result is analogous to the modern theory of critical phenomena, where a few qualitative properties of the system determine universal critical exponents. Both are fixed-point theories, and the number δ, he says, can be viewed as a critical exponent.

In 1979, at the suggestion of Pierre Collet (Ecole Polytechnique) and Jean-Pierre Eckmann (University of Geneva) that coupled differential equations with period doublings might show the same universal metric properties, Valter Franceschini and his collaborators at the University of Modena, Italy, did numerical studies that verified this hypothesis, for example, in two different truncations of the Navier–Stokes equation. In other words, they found that some continuous systems with many degrees of freedom behaved the same as one-dimensional systems.

Feigenbaum's theory has been cast in a more rigorous and abstract mathematical form by Collet, Eckmann[2] and Oscar Lanford (Berkeley). Essentially complete proof of Feigenbaum's result under certain assumptions was given recently by Lanford and also by Campanino (University of Rome), Henry Epstein (Institut des Hautes Etudes Scientifiques) and Ruelle.

Experiments. Last year Albert Libchaber and J. Maurer (Ecole Normale Superieure) did Rayleigh–Benard experiments[3] in liquid helium contained in a brick-shaped cell that was heated from below. As they increased the temperature difference between the top and bottom of the fluid, a bifurcation or change in period occurred. Libchaber and Maurer studied the state where two convective rolls are formed. A second bifurcation leads to an oscillatory instability. As the temperature difference was increased, the Fourier analysis of the temperature at one point in the fluid showed a sequence of subharmonic bifurcations (in which the frequency is halved) until the system reached turbulence. The experiment showed $2T$, $4T$, $8T$ and $16T$. With each bifurcation the interval in Rayleigh number became smaller so that by $32T$, Libchaber and Maurer were not able to distinguish the bifurcations experimentally.

When Feigenbaum learned of the Libchaber–Maurer experiment, he used part of their observed spectrum to compute[4] the rest of the spectrum. There are no free parameters in the calculation. Theory and experiment agreed to two significant figures. Experiments by Jerry Gollub, Steven Benson and Jethro Steinman (all then at Haverford College) also sometimes revealed[5] up to two period-doubling bifurcations. Feigenbaum has shown that in a period-doubling transition, the new peaks will be lower than the old ones by 8.2 db in intensity. This prediction is consistent with the Haverford experiment.

Very recently Michael Nauenberg and Joseph Rudnick (University of California, Santa Cruz) have developed a theory for predicting the height of the spectral peaks, too. Bernardo Huberman (Xerox Palo Alto Research Center) and Albert Zisook (University of Chicago) and Doyne Farmer (Santa Cruz) have shown that the integrated noise power spectrum in the chaotic regime grows as a power law.

Although the observation of the subharmonic route to chaos in a fluid has excited many workers, a real fluid can follow lots of other paths to turbulence, paths that are not as well understood theoretically. As Feigenbaum says, if the system does show period doubling, you can predict the rate at which it will do it. But at present there is no way to tell in advance what system will show period doubling.

Effects of external noise. Last year Guenter Ahlers (Santa Barbara) and Robert Walden (Bell Labs) did some Rayleigh–Benard experiments that they suggested were dominated by external stochastic processes. This suggestion has prompted consideration of the effects of external noise on period-doubling systems.

James P. Crutchfield (University of California, Santa Cruz) and Huberman did numerical experiments on period-doubling systems when they put in noise obtained from a random-number generator.[6] In this case they found that the number of doublings is finite instead of infinite. So chaotic behavior occurs earlier when external noise is present.

The Lyapunov exponent measures how fast two points in phase space separate in time. Huberman and Rudnick found[7] in systems without noise that the Lyapunov exponent scales universally near a point of infinite period doubling. That is, once the system becomes chaotic, the Lyapunov exponent grows with a power-law dependence, just like an order parameter in a phase transition.

Very recently three groups have studied the effects of external noise on the scaling of the Lyapunov exponents. Boris Shraiman, Eugene Wayne and Paul Martin (Harvard University) and Crutchfield, Nauenberg and Rudnick have found theoretically that in the presence of external noise, the Lyapunov exponent obeys a universal scaling law. A third group, Crutchfield, Farmer and Huberman, also did numerical experiments that produced the same results.

Martin told us the new work on noise clarifies the connection between the period-doubling route to chaos and the renormalization-group approach to phase transitions. In a phase transition, one varies the temperature to go from one phase to another. In the transition from a multiply periodic phase to a chaotic phase, one changes a control parameter, such as stress or Reynolds number. Because extended scaling laws apply in the period-doubling system, the noise there serves a role similar to an external magnetic field in a magnetic phase transition.

Applications. In 1979 Huberman and Crutchfield suggested that one might see solid-state turbulence in weakly pinned charge-density-wave systems and superionic conductors. Some groups are looking for evidence of such turbulence in $NbSe_3$, the fruit fly of charge-density waves. Huberman, Crutchfield and Norman Packard (Santa Cruz) have considered[7] the motion of a particle in a sinusoidal potential and found this system has chaotic solutions that reach chaos by period doublings. They believe the striking rises in noise observed in Josephson-junction oscillators may be due to period-doubling behavior.

Hyatt Gibbs, Fred Hopf, David Kaplan and Rick Shoemaker (University of Arizona) told us they have found evidence[8] for period doubling and optical turbulence in a hybrid optically bistable device, as predicted by Kensuke Ikeda (Kyoto University). The behavior of this one-dimensional system is in substantial agreement with that predicted by Feigenbaum, according to Gibbs.

Poincaré found that a Hamiltonian

system with two degrees of freedom is related to a transformation that preserves areas in two dimensions. Theorists have recently found that for these area-preserving systems, period doubling also occurs and universal behavior is found. The hope now is to extend the theory to Hamiltonians with more degrees of freedom. —GBL

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Surface EXAFS yields bond sites

Extended x-ray absorption fine structure (EXAFS), a powerful source of information about the immediate environment of selected atoms in systems without long-range order, is now being exploited for the study of surfaces. Using the high-intensity x-ray beams produced by synchrotron radiation from SPEAR, the e^+e^- storage ring at the Stanford Synchrotron Radiation Laboratory (SSRL), groups from Bell Labs[1] and Stanford[2] have recently been able to determine with high precision the bond lengths and adsorption sites of fractional monolayers of atoms adsorbed onto various substrate surfaces.

Such studies are intended to investigate the general nature of surface chemical bonding. The structural details of surface adsorption are also of practical importance for understanding chemical catalysis and surface oxidation. Furthermore, they can elucidate what goes on at interfaces in microelectronic heterostructures. Surface EXAFS (SEXAFS) is at present limited by signal intensity to the study of surfaces onto which foreign atoms or molecules have been adsorbed. But when higher intensity synchrotron radiation beams become available at dedicated storage rings now under construction, SEXAFS should be able to analyze surface structure in homogeneous materials.

Low-energy electron diffraction, the present standard technique for the study of surface structure, suffers from a number of limitations. Like Bragg x-ray diffraction, but unlike EXAFS, it requires long-range order in the material under study. Furthermore, the interpretation of LEED data is complicated by multiple scattering of the low-energy electrons. This requires fitting the data to mathematical models, and it limits the precision of bond length determination to about 0.1 Å.

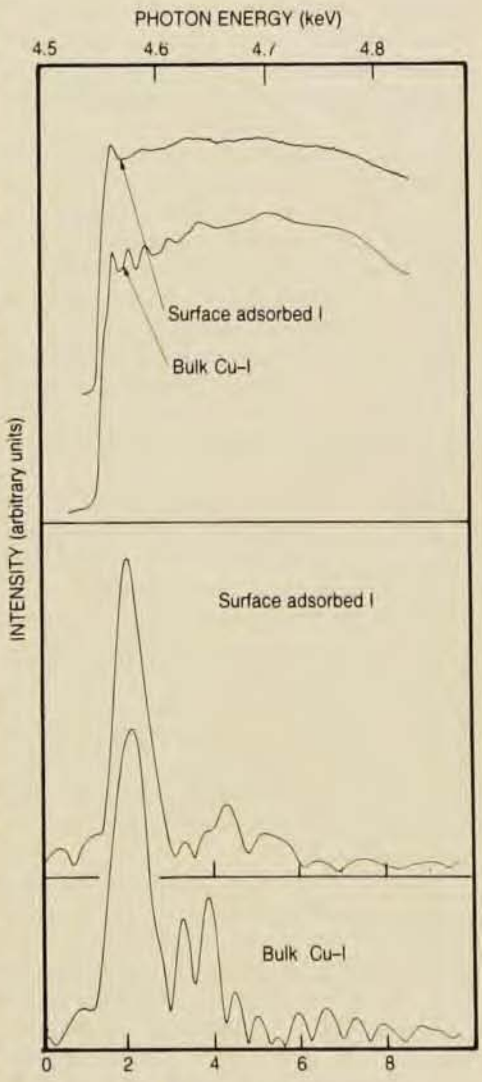
EXAFS. In the x-ray absorption spectra of atoms bonded in solids or molecules one frequently sees an oscillatory fine structure extending for several hundred eV above a particular absorption edge. R. de L. Kronig suggested a half century ago that this extended fine structure might contain information about the near neighbors of the absorbing atom. It was eventually understood that these EXAFS spectra result from interference between the photoelectron wave emerging from the absorbing atoms and the backscattered waves from elastic collisions of this photoelectron with neighboring atoms.

In 1971 Dale Sayers and Edward Stern (University of Washington), and Farrel Lytle (Boeing) proposed a method for determining the distances of these backscattering neighbors from the absorbing atom by Fourier analysis of the EXAFS spectrum (PHYSICS TODAY, October 1974, page 17). To investigate the immediate neighborhood of a particular atomic species in the sample, one looks at the EXAFS structure just above a convenient x-ray absorption edge for that species. The small oscillations in the x-ray absorption cross section, observed as one increases the incident energy, go like $\sin(2kR+\theta)$, where R is the distance from the absorbing to the backscattering atom, k is the photoelectron momentum, and θ is a small, energy-dependent phase shift. Thus if one can correct for the phase shift, the Fourier transform of the EXAFS spectrum will generate a number of peaks corresponding to the distances of nearby backscattering atoms. Furthermore, the amplitude of the EXAFS signal would be a measure of the number of neighboring atoms contributing to the backscattering.

With the advent of the intense synchrotron x-ray source at SPEAR (1974), and the demonstration that the phase shifts one needs to know are insensitive[3] to chemical environment (1976), EXAFS became a very precise tool for measuring specific bond lengths in bulk systems without long-range order, and in complex biological molecules. Bond lengths in such systems could now be measured to within a few hundredths of an angstrom. This high precision is very useful for choosing between alternative structural configurations.

Surface EXAFS. Being able to determine bond lengths with an accuracy almost an order of magnitude better than that afforded by LEED, EXAFS becomes very attractive for the investigation of surface structure. But an x-ray beam passing through even the thinnest feasible sample would encounter 10^5 more atoms in the substrate bulk than in the surface monolayer of interest. The surface EXAFS signal would thus be swamped by background from the substrate in a conventional EXAFS measurement, where the x-ray absorption cross section is measured simply by observing the attenuation of the beam passing through the sample.

What is needed is a surface-specific



SEXAFS spectrum for iodine adsorbed on (111) surface of copper crystal (top), shown with EXAFS spectrum of bulk CuI. The SEXAFS data were taken by Bell Labs group with x-ray beam polarized parallel to surface. Below are Fourier transforms of these spectra, after background subtraction. After filtering out high-frequency components and correcting for phase shift from bulk data, one gets I–Cu adsorbate bond length of 2.66 ± 0.02 Å.