The transition to turbulence

Modern optical and computer techniques and new concepts in the theory of nonlinear systems are yielding insights into such hydrodynamic instabilities as Couette flow, vortex streets and the Rayleigh–Bénard instability.

Harry L. Swinney and Jerry P. Gollub

Fluid flows have been studied systematically for more than a century and their equations of motion are well known, yet the transition from laminar flow to turbulent flow remains an enigma. The difficulty lies in the intractability of the nonlinear hydrodynamic equations that express the conservation of mass, momentum and energy for a fluid continuum. Although these equations can be linearized and readily solved for a system near thermodynamic equilibrium, the solutions of the nonlinear equations—required to describe fluids far from equilibrium—are generally neither unique nor obtainable.

Flow regimes

A fluid system can be driven away from thermodynamic equilibrium by imposing a gradient in the velocity, the density or the temperature. If there is a velocity gradient, the "distance" away from equilibrium is conveniently characterized by a dimensionless quantity, the Reynolds number, $R = VL/\nu$, where V and L are a characteristic velocity and length respectively, and ν is the kinematic viscosity, the ratio of the viscosity coefficient to the density. Similarly, in systems with density, temperature or other gradients, the distance away from equilibrium is described by dimensionless numbers proportional to the imposed gradient. We will use the term "Reynolds number" in a generic sense to refer to all such numbers.

For any fluid system sufficiently near equilibrium, there exists a unique, stable solution to the hydrodynamic equations. Regardless of the initial conditions, the fluid will approach this solution asymptotically for sufficiently small Reynolds number; however, as the Reynolds number is increased above some critical value R_c, this solution becomes unstable in the sense that an inverted pendulum is unstable to small fluctuations. A new flow pattern then develops. The solution for small Reynolds number reflects the temporal and spatial symmetry of the boundary conditions; in the transition to the second flow regime, this symmetry is generally broken.

Fluids in which the Reynolds number is increased beyond R_c often exhibit a sequence of distinct flow regimes. The increasingly complex flows display a variety of temporal and spatial patterns that are not in any obvious way related to the boundary conditions.

As the Reynolds number is increased further, these flows become irregular in space and show a chaotic or noisy time dependence. However, even the apparently chaotic flows are considered to be governed by the deterministic hydrodynamic equations. Although it seems paradoxical, a deterministic flow can appear chaotic; that is, as a consequence of the nonlinearity of the system, the equations can have exceedingly complex solutions that do not look as if they are the result of a deterministic process.

The round jet in figure 1 illustrates the transitions from laminar flow through an instability to chaotic flow. At what point can the name "turbulence" be applied to such a flow? Often this term is reserved for flows in the limit of very high Reynolds number, where the small-scale motions become essentially independent of boundary conditions. However, we will use the term here to describe any flow in which the dynamical variables exhibit noisy or chaotic time dependence. We make this choice because our viewpoint is that the essential distinction between laminar and turbulent flows is the presence of noise in the velocity field of a turbulent flow. Once this element is present, further increase in Reynolds number results in a gradual increase in the frequency and wavenumber scales of the velocity fluctuations.

In this article we will be concerned with the instabilities that precede turbulence rather than strongly turbulent flows. Although a century of investigation by talented theoreticians and experimentalists has produced a rather detailed understanding of the primary instability (at R_c) of various flows, the subsequent instabilities are still not understood. For more detailed discussions of hydrodynamic instabilities we refer the reader to a number of general works.[1]

What is the nature of these successive instabilities? Is there a limit to the number of instabilities that can occur before the flow becomes chaotic? Is there even a well defined value of the Reynolds number beyond which the flow becomes chaotic? These questions can not be answered with confidence in any single case, and we do not know to what extent there are universal answers to them.

Fluid flow is only one of many classes of nonlinear systems far from equilibrium that have largely resisted past attempts at understanding. Other examples include problems in population dynamics, chemistry, ecology, geophysics, plasma physics, biology and economics.[2] Ilya Prigogine, who was awarded the 1978 Nobel Prize in chemistry for his contributions to the thermodynamic theory of nonlinear systems far from equilibrium, has shown that a system taken away from equilibrium can become unstable and evolve a highly ordered structure. Examples of "dissipative structures" in chemical systems have been extensively

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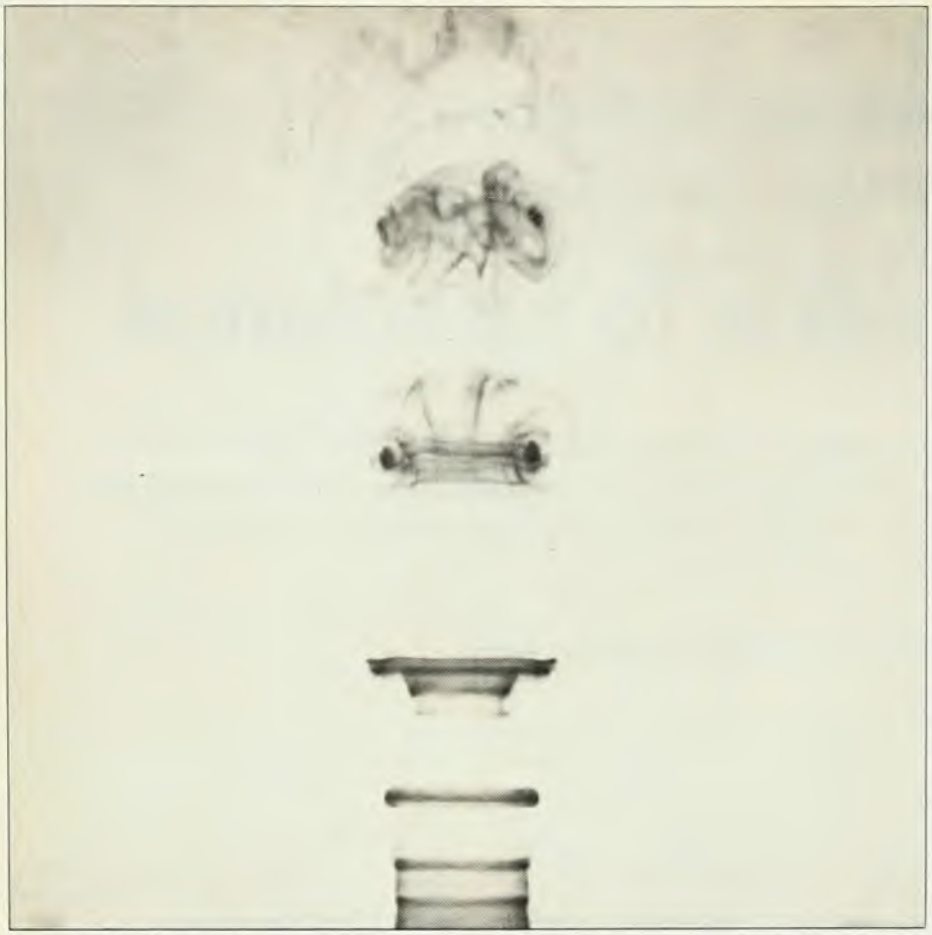


Instability in the shear layer of a round fluid jet. Starting with laminar flow, the fluid undergoes a shear instability and a transition to turbulence. What is the nature of such fluid instabilities? Photograph by Gene Bouchard, Stanford University. Figure 1

studied by Prigogine and his group.

The current wide interest in nonlinear systems far from equilibrium stems in part from an appreciation of their pervasiveness in many disciplines and in part from the hope that new developments in experimental techniques and in applied mathematics may lead to an improved understanding of them. In this paper we will describe several hydrodynamic systems and recent experimental and theoretical work that has begun to remove past roadblocks toward understanding them.

Three unstable flows

Experimental and theoretical efforts to understand hydrodynamic instabilities have concentrated on a small number of simple systems. Although there are many systems (including pipe flow and channel flow) that make a transition from simple laminar flow to turbulence without a sequence of well defined instabilities, we will consider three examples of systems that exhibit a sequence of hydrodynamic instabilities:

▸ The flow of a fluid contained between two concentric cylinders with the inner cylinder rotating, a geometry known as "circular Couette flow." In the initial laminar flow only the azimuthal component of the velocity is nonzero, but when a critical Reynolds number (proportional to the angular velocity of the inner cylinder) is exceeded, this flow becomes unstable and a new one is established. The structure of the new flow pattern is shown in figure 2a. This instability and the form of the secondary flow, which consists of a stack of toroidal vortices superimposed on the azimuthal flow, were observed and explained by Sir Geoffrey Taylor[3] in 1922 in a remarkable theoretical and experimental investigation. At a higher, well defined Reynolds number, Taylor vortex flow becomes unstable to transverse oscillations, and the vortices acquire travelling waves, as shown in figure 2b. At this point the flow is time-dependent, but strictly periodic. At much higher R, the flow becomes turbulent, in a process that is not well understood.

▸ The flow past a circular cylinder placed perpendicular to a uniform stream. In the laminar-flow regime, the streamlines vary smoothly around the cylinder; for larger R, the wake of the cylinder develops a striking travelling-vortex pattern known as a "Kármán vortex street," shown in figure 2c. Additional instabilities are observed before the flow ultimately becomes turbulent. For this geometry the analytic theory is particularly difficult and has not been extended beyond the simple laminar-flow regime.

▸ The instability in a fluid contained between two horizontal thermally conducting flat plates with the lower one warmer than the upper one. For this system with an imposed temperature rather than velocity gradient the unique solution of the hydrodynamic equations near equilibrium corresponds to pure conduction, with a linear temperature gradient and *no* flow. The distance away from equilibrium R is given by the Rayleigh number, which proportional to the temperature difference between the plates. In a classic paper on hydrodynamic stability Lord Rayleigh[4] showed in 1916 that the pure conduction state would become unstable against a fluctuation in the form of two-dimensional convection rolls, as shown in figure 2d. This instability is now known as the Rayleigh–Bénard instability, in recognition of Henri Bénard's observations of convection cells in experiments in 1900 on the related problem of convection in a fluid with a free surface. As R is increased beyond R_c in the Rayleigh–Bénard system, a three-dimensional pattern occurs if the viscosity is sufficiently high, and at larger R the convection becomes time-dependent (see figure 2e). The transitions in Rayleigh–Bénard convection beyond the primary instability depend on another dimensionless number, the Prandtl number σ, which is the ratio of the kinematic viscosity to the thermal diffusity. For figure 2d, $\sigma = 100$; figure 2e, $\sigma = 63$, and for figure 5 (right), $\sigma = 2.5$.

There are many other instances of hydrodynamic instabilities, usually driven by some combination of gradients of temperature, density or velocity. For example, the regular cloud patterns known as cloud streets are probably produced by an instability involving both density and velocity gradients. Instabilities can also be produced in conducting fluids or plasmas by electric and magnetic fields.

Experimental techniques

Because the theory of fluids far from equilibrium is difficult and the solutions not intuitive, experiments play a crucial role in the guidance of theoretical developments. This has been particularly true in the study of hydrodynamic instabilities, where markers such as dyes, bubbles and smoke have often revealed unexpected secondary flow patterns.

A great variety of markers has been developed to make flows visible. Anisotropic particles make it possible to observe the overall structure of a flow, as in figure 2. Electrochemical techniques are used to induce color changes or bubble formation. The photographs can sometimes be analyzed quantitatively, but different types of markers do not in general respond to the same dynamical features of the flow, and photographs do not indicate reliably whether or not a flow is chaotic. Despite these limitations,

flow-visualization techniques continue to be an important tool for surveying the possible flow patterns in a system. Furthermore, they are often used to complement other, more quantitative, techniques that measure local properties of a flow.

Quantitative methods for studying instabilities include measurements of bulk properties that reflect changes in the flow, such as the torque in rotating flows, the mass flux in pipe flow and the heat flux in convection. These methods can be quite sensitive and are even useful in studying time-dependent phenomena.

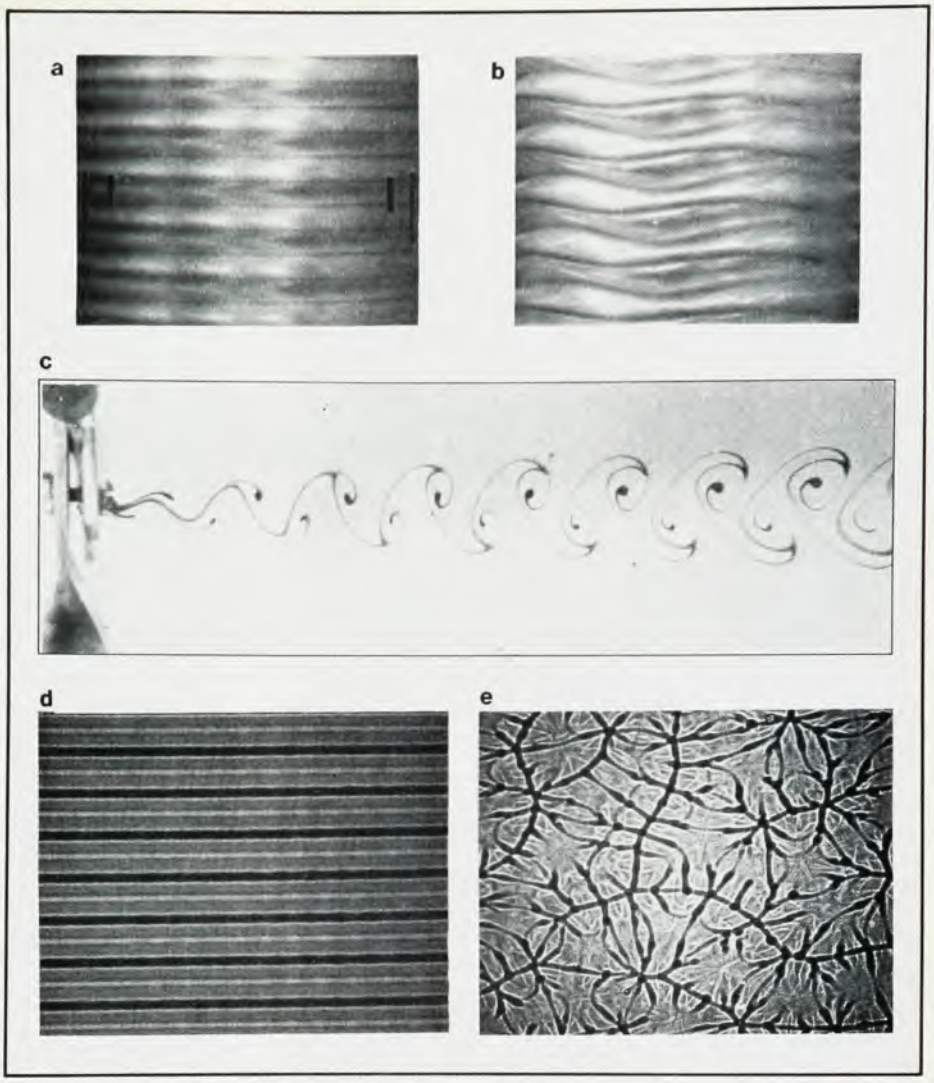
The principal quantity of interest in flow studies is of course the velocity field. The two major techniques for measuring the local velocity are hot-wire anemometry and laser Doppler velocimetry.

In hot-wire anemometry the fluid velocity is deduced from measurements on a very fine wire with a resistance that depends on the rate of heat transfer to the fluid moving past it. The technique has been highly refined, and it is possible to determine velocities in regions as small as 0.1 mm in diameter, and to follow velocity fluctuations up to about 50 kHz. However, the method has two disadvantages in addition to the limited frequency response:

- the technique is not absolute, making calibration of the wires necessary, and
- a probe must be inserted into the flow, the wake of which itself can disturb the flow.

In the laser Doppler technique the fluid velocity is determined from measurements of the Doppler shift of light scattered by the moving fluid.[5] As illustrated in figure 3, the scattered light is mixed with light from the laser on a square-law photodetector, and the resultant photocurrent has a component oscillating at the Doppler-shift frequency ν_D. The component of the velocity V parallel to the scattering vector $\mathbf{k} = \mathbf{k}_0 - \mathbf{k}_s$ is given by $V = (\frac{1}{2}\lambda/\sin\frac{1}{2}\theta)\nu_D$, where θ is the scattering angle and λ is the wavelength of the light. Any velocity component can be selected by appropriate choice of geometry, and measurements can be made of Doppler shifts ranging from about 1 to about 10^8 Hz (corresponding approximately to velocities from 10^{-4} to 10^5 cm/sec). The typical linear dimensions of the scattering volume are about 0.1 mm. Usually ν_D is much greater than the characteristic frequencies of the fluid motion; therefore measurements of ν_D in short time intervals yield essentially the instantaneous velocity $V(t)$. The advantages of laser Doppler velocimetry are that it is direct, absolute and nonperturbative. The major disadvantage is that the equipment is complex and expensive.

The laser Doppler technique can be used to produce detailed contour maps of the velocity field in steady flows, as shown in figure 4. This map was made by automatically scanning a small convection cell with respect to the scattering volume in two dimensions, and then using a computer to construct contours of a constant velocity component. We can see immediately that the cell contains two convective rolls.



Some examples of instabilities in fluid flows. The fluid in photo **a**, trapped between an inner rotating cylinder and an outer fixed cylinder, exhibits time-dependent toroidal vortices known as "Taylor vortices." At greater angular velocity the fluid (**b**) exhibits travelling waves superimposed on the Taylor vortices. Flowing past a cylinder, the fluid in **c** exhibits another instability, a so-called "Kármán vortex street." The two-dimensional convection rolls in **d** form a layer of fluid between two horizontal plates, the lower one being warmer. A more complex pattern develops for greater temperature differences, as shown in **e**. The photographs are by R. Fenstermacher (a and b), D. J. Tritton (*Physical Fluid Dynamics*, Van Nostrand Reinhold, New York, 1977) (**c**), and F. Busse and J. Whitehead (**d** and **e**). Figure 2

By interfacing a hot-wire or laser Doppler anemometer to a computer, long records of a velocity component $V(t)$ can be accumulated for Fourier analysis. This is important in studies designed to determine whether a flow is singly or multiply periodic, or whether it is chaotic. The power spectrum $P(f)$ (magnitude squared of the Fourier transform) of $V(t)$ contains only sharp peaks if the flow is periodic, but it contains broad-band spectral features if the flow is chaotic. Hence the frequency resolution of the spectrum is of critical importance. If no averaging is employed to reduce statistical noise, the frequency resolution is approximately $2/T$, where T is the duration of the data record. The maximum frequency in a spectrum is $(2\Delta T)^{-1}$, where ΔT is the interval between adjacent samplings of the velocity. Because both high resolution and a broad spectral range are needed to distinguish between the different dynamical regimes of a flow, data records should contain as many samples ($n = T/\Delta T$) as possible.

Laser Doppler experiments

Detailed quantitative studies of sequences of instabilities leading to turbulence are under way in a number of laboratories.[6] As an example of such experiments, we will describe the laser Doppler studies of circular Couette flow and Rayleigh–Bénard convection made in our laboratories.[7] These experiments were designed to see whether different systems exhibit similar dynamical regimes as they are driven further from equilibrium.

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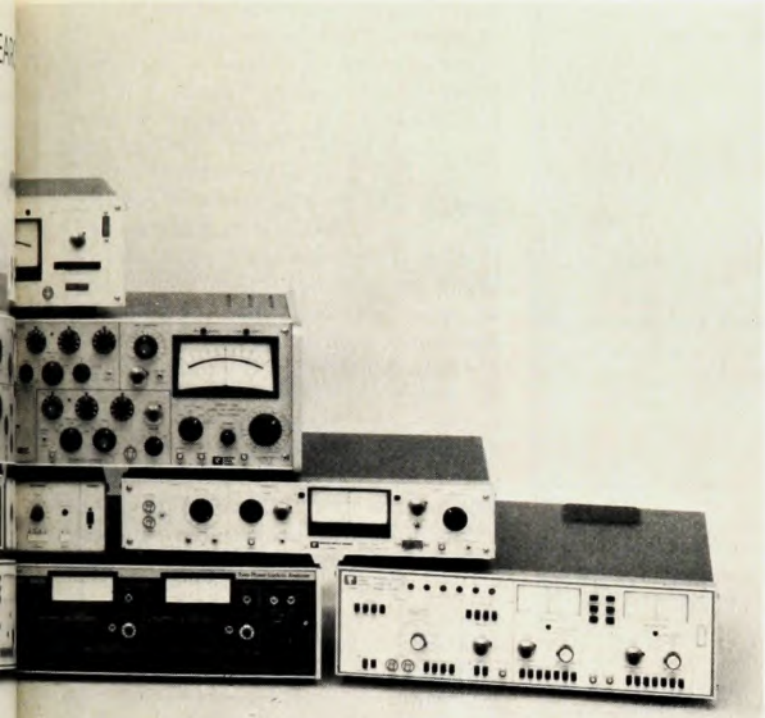

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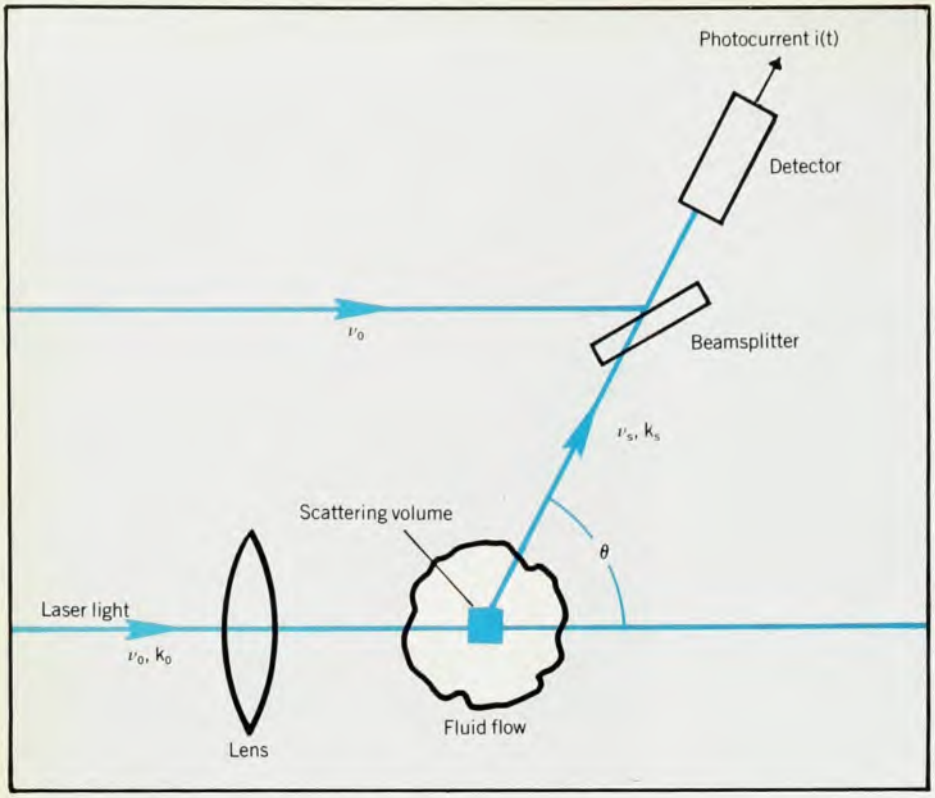


Laser Doppler system for velocity measurement. Laser light of frequency ν_0 and wavevector $\mathbf{k}_0$ is focussed on the fluid. The scattered light from a small volume, with its frequency Doppler shifted to ν_s, passes through a beamsplitter and is mixed with the unshifted light. The alternating component of the photocurrent $i(t)$ then oscillates at the Doppler frequency $\nu_D = \nu_s - \nu_0$. Velocities of 10^{-4}–10^5 cm/sec can be measured with this technique. Figure 3

V_r, was measured in a Couette-flow system for Reynolds numbers ranging from that corresponding to the onset of Taylor vortex flow up to turbulent flow. In Taylor vortex flow, V_r is time-independent but has a periodic variation in the z (axial) direction. The x-dependence of V_r is described by a single Fourier component and its harmonics; the harmonic content increases with increasing R. For time-independent Rayleigh–Bénard convection, Pierre Bergé and Monique DuBois have observed similar behavior.[6]

As R is increased (for example, by increasing the rotation rate in Couette flow or the temperature difference in convection) the flows begin to oscillate in time, and the velocity measured at a fixed point in a sample has a spectrum like those shown in the upper curves in figure 5. In the Couette system the oscillations represent the passage of the waves seen in figure 2b by the point of observation. The spectra for both systems contains no frequency components other than the fundamental and its harmonics, and these components have no width beyond the instrumental width ($\Delta f/f$ is less than about 10^{-3}). The flow in this regime thus is strictly periodic, with the oscillations coherent over hundreds of cycles.

As R is increased for these two systems, the close correspondence between them persists. A second time-dependent instability occurs in each system, resulting in a flow with two incommensurate frequencies, as shown in the middle curves in figure 5. Such a system is termed "quasiperiodic." Although the velocity spectra appear rather complex, each consists of only two fundamental frequencies and their linear combinations. These quasiperiodic states are not noticeable in flow visualization, and have not been explained theoretically.

As R is increased further, broad-band noise components appear in the spectra in addition to the narrow peaks (see the lower curves in figure 5). These transitional states have some properties of *both* periodic and turbulent flows: The narrow peaks indicate that the velocity correlations persist for at least as long as the experiment duration, yet the broad components clearly indicate a chaotic element in the flow.

Eventually, at large R, the sharp spectral components disappear and the transition to turbulence is complete. A large interval in Reynolds number or Rayleigh number is contained between onset of nonperiodicity (broad-band spectral components) and the loss of the remaining sharp structure. It is important to understand that the existence of some randomness in the time dependence of the dynamical variables after the sharp components have disappeared does *not* imply a totally featureless flow. The Taylor vortices and Bénard rolls persist in the sense that time averages of the velocity field would reveal their presence even at very high Reynolds and Rayleigh numbers. This persistence of such so-called "large-scale structures" in turbulent flows is very common. They occur in shear layers, wakes and boundary layers, and they are believed to be responsible for some of the noise of jet aircraft.

The sequence of instabilities leading to turbulence is quite similar in these two systems. Notably, there are distinct periodic and quasiperiodics states, and the number of instabilities preceding the onset of broad-band noise is quite small. However, this pattern is certainly not the only possible one. As we have mentioned, many flows make a direct transition from the laminar flow to turbulence without passing through a sequence of instabilities. Even in the Couette and Rayleigh–Bénard geometries the transitional behavior can be quite different from the one we have described when a geometrical parameter, such as the ratio of the cylinder radii is altered. For example, Guenter Ahlers and Robert Behringer[6] of Bell Laboratories discovered that in Rayleigh–Bénard convection in fluid containers with a relatively large ratio of diameter to height (= 57), broad-band noise occurs in the heat flux just above the onset of convection; there is no periodic regime at all. This dependence of even the qualitative features of the transition process on a geometrical variable is surprising, and does not offer much encouragement for efforts to construct universal models for the transition to turbulence.

Stability theory

The primary instability of a flow can be determined by a technique known as linear stability analysis.[1] This method deals with the effect of a small fluctuation away from the basic flow that is unique and stable for sufficiently small R. The analysis is in terms of normal modes, which constitute a complete set of eigenfunctions. For example, in circular Couette flow a fluctuation in the radial velocity is expressed as a superposition of Fourier components of the form

$$A(t)f(r)\cos(kz + \psi)$$

where $f(r)$ is a function of radial position. The hydrodynamic equations are linearized in the amplitude of the fluctuation, and the linearized equations are solved to determine whether the fluctuation decays or grows in time for a given R. For small perturbations the time-dependent amplitude has an exponential time dependence,

$$A(t) = A_0 e^{\gamma t}$$

and an algebraic equation for $\gamma(k,R)$ is obtained. Its solution determines those values of the Reynolds number and wavenumber k for which the perturbation will grow, as determined by the condition that the real component of γ is positive, as indicated in figure 6. For $R < R_c$,

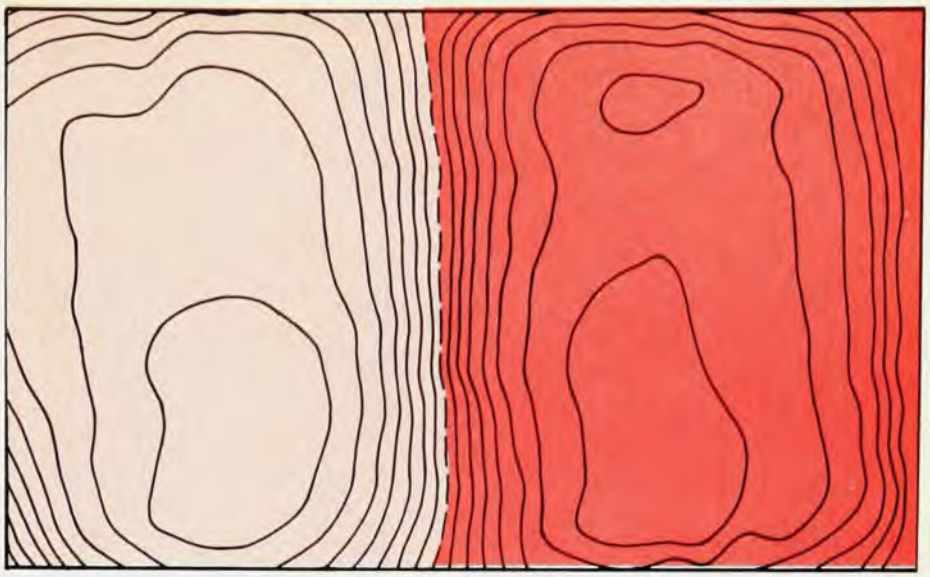
$\mathrm{Re}(\gamma) < 0$ and the perturbation can not grow for any wavenumber. At $R = R_c$ a mode with $k = k_c$, corresponding to the Taylor vortices, is neutrally stable, with $\mathrm{Re}(\gamma) = 0$; all other modes decay. If R is just above R_c, growth can occur for a narrow range of wavenumbers about k_c.

This analysis for Couette flow determines the numerical values of R_c and k_c in terms of the ratio of radii of the cylinders confining the fluid; similar analyses have been done for many other types of instabilities. For some of them the parameter γ has a nonzero imaginary part, which simply indicates that the instability results in a periodic oscillation of the flow field. In some systems, however, linearized analysis is not appropriate for the study of even the primary instability, because they are stable against *infinitesimal* fluctuations yet unstable against *finite* fluctuations; the analysis of such instabilities is obviously more difficult and generally needs a nonlinear theory.

The linear analysis just described determines the critical values of the Reynolds number and the dominant normal mode, but does not describe the finite-amplitude regime that is reached asymptotically as $t \to \infty$. To accomplish this it is necessary to use the nonlinear dynamical equations. Normal-mode decomposition is still a useful technique, but the expansion must be truncated to obtain a solvable finite set of coupled equations in the amplitudes of the various normal modes. Sufficiently near the neutral-stability curve these equations can be solved, with the following typical result.

After a period of exponential growth, the amplitude of the fundamental mode with wavenumber k_c saturates at a value proportional to $\epsilon^{1/2}$ for small ϵ where $\epsilon = (R - R_c)/R_c$. Harmonics with wavenumbers that are integral multiples of k_c will also have finite amplitudes, because the nonlinearity couples the various modes. However, the amplitudes of these depend on larger powers of ϵ, so that they become substantial in comparison to the fundamental mode only when R_c is substantially exceeded. These predictions have been quantitatively confirmed by means of light-scattering methods for both the Rayleigh–Bénard and Taylor instabilities.

By this time the reader should be in a position to appreciate the difficulty of extending this analysis to describe secondary and tertiary instabilities. The base flow about which the next perturbation analysis must be performed is known only as an infinite sum of interacting modes. The stability of this solution with respect to arbitrary perturbations can be analyzed, and a set of coupled linear ordinary differential equations is obtained. This set constitutes an eigenvalue problem for the growth rate of the secondary instability. This eigenvalue can be determined numerically to obtain the neutral-stability curve of the secondary instability,[9] but it has rarely been possible to extend this analysis into the nonlinear regime above the onset of the secondary instability for hydrodynamic flows of interest. The methods of nonlinear stability theory are too cumbersome to give a rigorous explanation of a sequence of even two instabilities. The possibility of giving a rigorous explanation of nonperiodic flow by an extension of this method appears remote.



Contour map of the velocity field in the horizontal plane of a fluid in convection, obtained by laser Doppler scanning under computer control. The region shown is 14 × 24 mm; the contours are separated by 100 Hz in Doppler shift, corresponding to 0.5 mm/sec in velocity in increment. The measured velocity component, directed parallel to the long axis of the figure, has opposite signs to the left and right of the line for zero velocity, shown as a dashed line. Figure 4

How turbulence starts

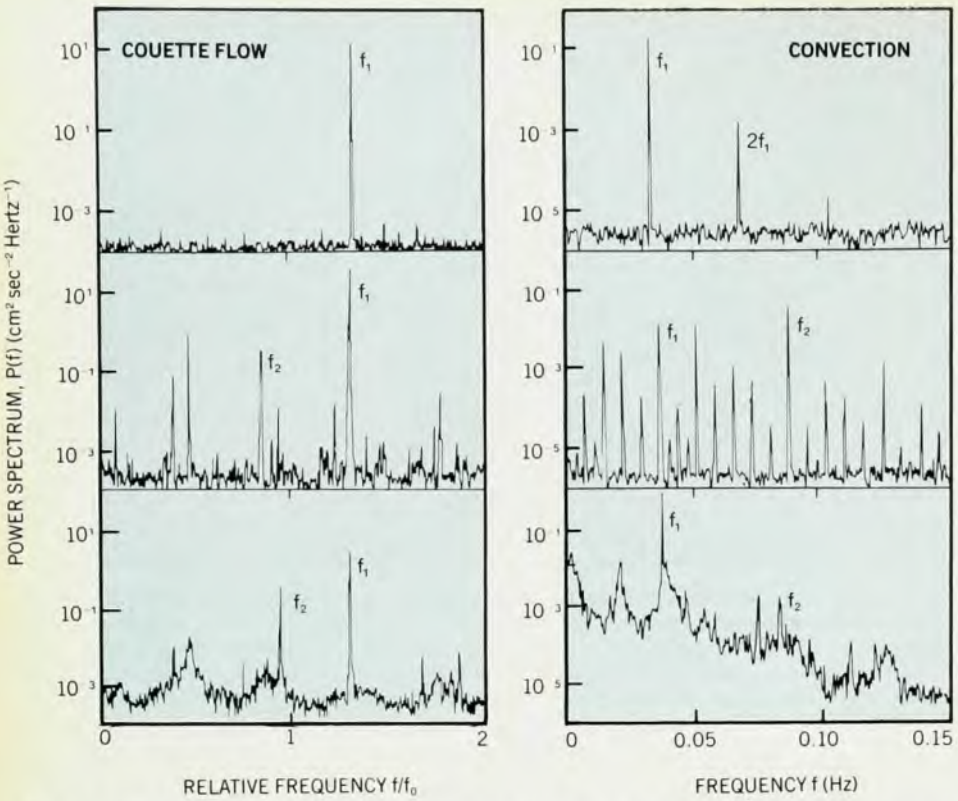
Even though it is difficult, because of practical obstacles, to treat sequences of instabilities by stability theory, we might still ask what the results would be qualitatively if the calculation could be carried out. In 1944 Lev Landau[10] suggested that an infinite sequence of instabilities would occur, each adding a new frequency to the motion. Turbulence, according to Landau, could be identified with a motion consisting of a superposition of so many frequencies that it is "complicated and confused." In his view there is no well defined onset of turbulence. Even at large R, where the flow appears quite complicated, it is assumed to be a superposition of periodic modes with generally incommensurate frequencies—a quasiperiodic rather than a chaotic flow.

An alternative view of turbulence was suggested by Edward Lorenz in a 1963 paper entitled "Deterministic Nonperiodic Flow".[8] A numerical study of a nonlinear model system (described in the following section) led Lorenz to suggest that turbulent flow is genuinely nonperiodic rather than quasiperiodic. Lorenz discussed the implications of the unpredictability of nonperiodic flow on long-range weather forecasting. His paper escaped the notice of most physicists and applied mathematicians for a decade, but it has subsequently had a profound effect on the theoretical development of the dynamics of nonlinear systems.

The qualitative difference between *nonperiodic* and *quasiperiodic* behavior should be emphasized. The essential feature that distinguishes a nonperiodic flow from a quasiperiodic flow is the sensitivity of the former to initial conditions: In a quasiperiodic regime, two flows with nearly identical initial conditions will remain nearly identical for all times, but in a nonperiodic regime the flows will ultimately evolve quite differently, no matter how nearly close the initial conditions were. Tien Yien Li and James Yorke[11] used the term "chaos" to describe the erratic behavior characteristic of nonperiodic regimes of nonlinear systems; chaos is now widely used in this sense (and used it so in this article). Whereas no fluid flow has been proven to be chaotic, a classical gas composed of hard spheres has been rigorously proven to be chaotic.[12]

A 1971 paper by David Ruelle and Floris Takens,[13] "On the Nature of Turbulence," has also contributed substantially to discussions of the transition to turbulence. Using considerations based on the phase-space topology of solutions to the dynamical equations, Ruelle and Takens argue that for most flows the nonlinear interactions would produce chaotic solutions following three or four instabilities. They use the term "strange attractor" to describe the limit set, in phase space, of these chaotic solutions.

How do experiments compare with the ideas of Landau, Lorenz, and Ruelle and Takens? Long sequences of instabilities resulting in quasiperiodic motions with



Power spectra of a component of the velocity of fluids in time-dependent Couette flow (left) and Rayleigh-Bérnard convection (right). Different dynamical regimes are observed with increasing Reynolds number. The upper curves correspond to periodic flow, the middle ones to quasiperiodic flow with fundamental frequencies f_1 and f_2, and the lower curves to chaotic flow containing both discrete frequencies and broad-band noise. In the data on the left, f_0 is the frequency of rotation of the inner cylinder of the Couette system (pictured in figures 2 a and **b**); the inner and outer radii are 22.2 and 25.4 mm, the height is 62.5 mm. The convection cell is rectangular, with a base 16 × 28 mm and 8 mm in height. The Couette-flow spectra were obtained by R. Fenstermacher and the convection spectra by S. Benson. Figure 5

many frequencies, as envisioned by Landau, have never been observed. However, there have been few experiments that could distinguish between genuinely chaotic flows and complex quasiperiodic ones, a distinction that is profound mathematically but difficult to make experimentally. Nevertheless, although it appears clear that flows generally become chaotic after a small number of instabilities, as Ruelle and Takens suggested, substantial differences mark the transition processes of different systems. These differences can not be predicted by a universal model.

The Couette and convection flows we have described become nonperiodic with the first appearance of broad spectral features, but these come in so gradually that most of the spectral energy initially remains in the sharp spectral components. Only at much larger R is the flow strongly chaotic in the sense that all of the spectral energy is broad-band, and only then is the flow turbulent in the sense generally understood by workers in fluid dynamics. Even this flow should perhaps be termed weakly turbulent; it is certainly not fully developed, homogeneous, isotropic turbulence.

This distinction between genuinely chaotic flows and complex quasiperiodic ones may be irrelevant to understanding *some* features of strongly turbulent flows. In statistical mechanics the averages of macroscopic quantities can be calculated, without knowledge of the detailed time evolution of the system, by introducing certain statistical assumptions. Similarly, in hydrodynamics it is possible to calculate statistical averages such as the position dependence of the mean velocity, but only by supplementing the equations with additional assumptions. However, for these assumptions to be valid it is usually necessary for the flow to be strongly turbulent. A deterministic, nonstatistical approach may be more appropriate for laminar and weakly turbulent flows.

Finite models

Accurate theoretical models of nonlinear dynamical systems are generally impossible to solve analytically, but some workers are using numerical techniques to study nonlinear models with a few degrees of freedom. Although these models are often highly simplified, they may exhibit the qualitative behavior of real systems. The classic prototype for such studies is the three-variable model whose chaotic dynamics Lorenz discovered.

The Lorenz model had its origin in Rayleigh–Bénard convection. The two-dimensional roll pattern observed above the critical Rayleigh number is described by two velocity components and by the deviation of the temperature from a linear conduction profile. These three fields were expanded in two-dimensional Fourier series and substituted into the hydrodynamic equations. Keeping only the lowest-order terms, Lorenz obtained the following equations:

$$\frac{dx}{dt} = \sigma(y-x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = -bz + xy$$

where $x(t)$ is proportional to the amplitude of the convective motion, $y(t)$ is proportional to the temperature difference between the ascending and descending currents and $z(t)$ is proportional to the deviation of the temperature profile from linearity. The constants in the equations are the Prandtl number σ mentioned above, the Rayleigh number in units of its critical value, r, and a constant, b, related to the wavenumber of the fundamental node.

The equations of the Lorenz model represent an extremely severe truncation: An infinite set of coupled ordinary differential equations has been reduced to three equations. The model therefore cannot be a realistic one for Rayleigh–Bénard convection significantly beyond $r = 1$. However, the Lorenz model is of intrinsic interest because of its fascinating mathematical properties. With the values of σ and b chosen by Lorenz, the system of equations has a stable solution $x = y = z = 0$ for $r < 1$, and two stable solutions,[8,14,15]

$$x = y = \pm[b(r-1)]^{1/2},\ z = r - 1$$

for $1 < r < 24.1$. In the "phase space" spanned by the amplitudes x, y and z (not to be confused with the spatial coordinates), the state of the system at any time is given by the point $[x(t), y(t), z(t)]$. For $r < 1$, all phase trajectories (representing different initial conditions) asymptotically approach a single point, the origin. Similarly, if $1 < r < 13.9$, all trajectories asymptotically approach one of the two stable solutions (which correspond to left- and right-handed rolls).

Between 13.9 and 24.1, James Kaplan and his associates[15] have found a complex transitional behavior; there is a metastable state, which can appear chaotic for arbitrarily long times, but the system finally settles down to a steady state. For $r > 24.1$, all trajectories are attracted toward a subspace on which they wander erratically *ad infinitum*. This is an example of a strange attractor. Many scientists are now investigating the properties of strange attractors for different nonlinear systems.

The Lorenz model demonstrates that chaotic dynamics can be inherent in

deterministic equations; thus the model supports the hypothesis that the complex dynamical behavior called turbulence is inherent in the hydrodynamic equations rather than being caused by random influences or a breakdown of the equations. Moreover, the Lorenz model shows that in a strongly nonlinear system *only three* degrees of freedom are needed to produce chaos! It has been conjectured that chaos in a system with many degrees of freedom (such as a fluid flow at large Reynolds numbers) is not qualitatively different from chaos in a system with a few degrees of freedom. Investigations of nonlinear model systems with a few degrees of freedom are now under way in many areas of science.[2] The hope is that such studies will lead to general insights into the chaotic dynamics of nonlinear systems.

Although there is a real possibility that models consisting of only a small number of coupled modes can explain the qualitative features of the onset of nonperiodicity in systems such as circular Couette flow and Rayleigh–Bénard convection—particularly when the geometry itself constrains the number of accessible modes—quantitative efforts to explain actual experimental data such as those of figure 5 by dynamical models with a small number of modes have been limited to date. One noteworthy effort to explain the transition to turbulent convection at low Prandtl number was undertaken by John McLaughlin and Paul Martin.[14] They constructed a model in the spirit of Lorenz, carefully selecting a set of 39 coupled Fourier modes. This model shows both periodic and chaotic states for reasonable values of the parameters of the problem. The calculations were not extensive enough to determine whether quasiperiodic states were also present.

New tools

It is becoming clear that progress towards understanding the transition to turbulence has been limited in the past because both the mathematical concepts and the experimental methods have been inadequate to cope with the complexity of the problem. The hypothesis that chaotic fluid flow may be represented as a special type of attractor in a dynamical system depended for its formulation on recent advances in topology and the theory of differential equations. The ability to distinguish experimentally between chaotic and quasiperiodic flows depended on such modern experimental methods as laboratory computers, lasers and sophisticated electronic signal processing. Perhaps these and other new theoretical and experimental tools will ultimately yield at least a qualitative explanation of the origins of chaotic fluid flow.

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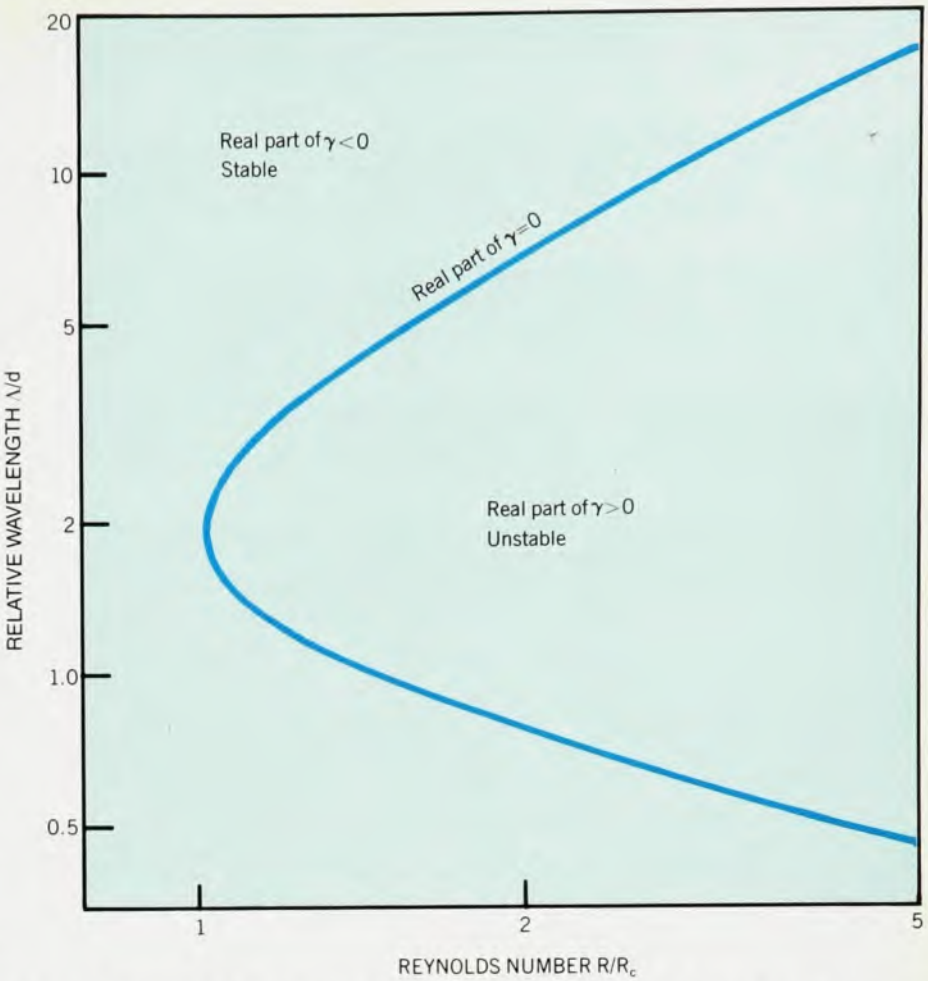

The curve of neutral stability for Taylor-vortex flow, as calculated from a linear stability analysis (S. Chandrasekhar, reference 1, page 303). The wavevector k is related to the axial wavelength Λ by $k = 2\pi/\Lambda$; d is the gap between the cylinders. Figure 6

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