The laser instability

Exploring the analogy of the lasing transition to such critical phenomena as ferromagnetism and the onset of convection in fluids leads to a clearer picture of the behavior of cooperative systems.

Vittorio Degiorgio

Many physical systems exhibit a transition from a disordered state to an ordered one following the change of an externally controlled parameter. Among these cooperative phenomena are the phase transitions of matter, the development of a convection pattern in a fluid layer heated from below—and the onset of lasing in a laser. Ordering phenomena of systems in thermal equilibrium have been known for a long time; their properties near the transition point were studied intensely in the last decade. A wealth of experimental data has shown marked similarities among what appear to be very different phase transitions: A fluid near its critical point behaves very much like a ferromagnet near its Curie point, or to a binary liquid mixture near the consolution point. To see these similarities we must make the appropriate choice for the corresponding variables. A considerable amount of work has been devoted to explaining the simple, universal behavior of thermodynamic systems in the region of the critical point.

Recently the concepts and techniques introduced to describe second-order phase transitions were shown to be applicable also to the study of cooperative phenomena in systems far from thermal equilibrium. A close analogy is found between the results of the laser theory and those of the Landau approach to phase transitions.[1,2] Instabilities in some electronic devices likewise show analogies similar to that discussed for the laser.[3] Furthermore, remarkable similarities have been discovered between the instabilities of lasers and fluids. Starting from these facts, experimenters have performed several investigations of instabilities, and attempts toward a unified theory of instabilities in open systems have been pursued. These studies consider, in particular, the conditions under which the macroscopic equations describing the evolution of the order parameter near threshold take the same structure as the laser equations.[4]

The list of cooperative phenomena bearing some connection with a phase-transition-like description is a long one, and includes examples from chemistry, biology and sociology.[4,5] However, to clarify the physical basis of the problem in this article, I will place particular emphasis on a few systems—primarily the laser, for which the analogy has already produced meaningful quantitative results.

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An ordered state

A laser consists typically of a dilute collection of active atoms in an optical cavity, which is excited by an external source of energy; see figure 1. If the pump power fed into the cavity is weak, the laser acts as an ordinary lamp. However, if both the active atoms and the external energy source are chosen properly, there is a critical pump power above which the device will emit a highly directional optical beam with a very long coherence time. The threshold value of the pump power is determined by the characteristics of the active atoms and by the energy losses of the optical cavity.

This behavior of the laser can be seen as a transition from a disordered state below threshold, where the atoms emit waves with random phases independently of each other, to an ordered state above threshold, where stimulated emission is predominant.

From this short description the concept emerges that the laser is not a system in thermal equilibrium but rather one in which a stationary ordered state is created and maintained by an energy flow through the device. There is therefore no *a priori* reason why the concepts and techniques developed in equilibrium thermodynamics to study phase transitions should be useful in the laser case.

Indeed, the properties of the laser have been described by a treatment that starts from a microscopic approach and takes the specific features of the problem directly into account.[6,7] Particular efforts have been devoted to the study of the region around threshold, because it is the behavior in this region that provides us with a better understanding of the laser transition. From the experimental point of view, it was a very fortunate coincidence to have available a specific system, the helium–neon laser, in which the transition region is rather easily accessible for accurate measurements. The investigation of the threshold region revealed many features qualitatively similar to those encountered in second-order phase transitions near their critical points, such as a large increase both in the magnitude and in the decay time of amplitude fluctuations of the laser field.[8]

The analogy was put on a quantitative basis by comparing the laser theory with the so-called "mean-field" (or Landau) treatment of second-order phase transitions. The origin of this similarity becomes evident when we recall that the usual treatments of laser behavior are self-consistent-field theories. In the laser analysis each atom is a radiating dipole in the electromagnetic field emitted by all the other atoms. The radiation field produced by this set of radiating atoms is then calculated in a self-consistent fashion. The physics of the laser is similar in this way to that of a ferromagnet in which each spin sees a mean magnetic field due to all the other spins and aligns itself accordingly, thus adding its contribution to the average magnetic field.

The laser instability

The laser can be schematized, as any oscillator can, as a positive-feedback amplifier. The gain at optical frequencies is provided by the pumped active medium and the positive feedback is obtained by the two parallel mirrors that form the optical cavity.

This cavity can sustain many modes. However, only the mode with the largest net gain will oscillate in the region near threshold. Because we are mainly interested in this region, our discussion will be confined to a single-mode laser.

The laser electric field can be written as $\mathbf{E}(t)e^{i\omega t}$, where ω is the angular frequency of the laser mode and $\mathbf{E}(t)$ is a complex quantity with temporal variations much slower than those of the oscillating exponential factor. The dynamical response of the system is well described by the following equation of motion which expresses the competition between the gain g and the losses γ,

$$\frac{d\mathbf{E}}{dt} = (g(\mathbf{E}) - \gamma)\mathbf{E} \qquad (1)$$

Near threshold the laser gain can be written as $g(E) = g_0 - \beta E^2$, where $E = |\mathbf{E}|$ and both g_0 and β are proportional to the unsaturated population inversion σ, the difference between the number of atoms in the upper laser level and that in the lower level. The threshold condition, $g_0 = \gamma$, defines the threshold population inversion σ_t.

The steady-state behavior of the laser field is easily derived from equation 1. Below threshold ($\sigma \leq \sigma_t$) the only solution is $E = 0$. Above threshold $\sigma > \sigma_t$ the only stable solution is

$$E = E_0 = A\left(\frac{\sigma - \sigma_t}{\sigma}\right)^{1/2} \qquad (2)$$

where A is a proportionality constant. Note that only the modulus of the field is assigned; the phase ϕ is arbitrary.

The derivation of the steady-state solution of equation 1 is shown graphically in figure 2. The unsaturated population inversion σ can be considered a state variable for the laser only below threshold; above threshold the steady-state population inversion is locked to the threshold value σ_t. This results directly from the fact that the laser model introduces linear losses and saturable gain.

Hermann Haken noted that equation 1 is formally similar to the approximate equation of motion of an overdamped anharmonic oscillator.[4] This suggested defining a kind of laser "potential"

$$G(E) = -\frac{g_0 - \gamma}{2}E^2 + \frac{\beta}{4}E^4 + \text{const} \qquad (3)$$

This laser potential, along with the equation

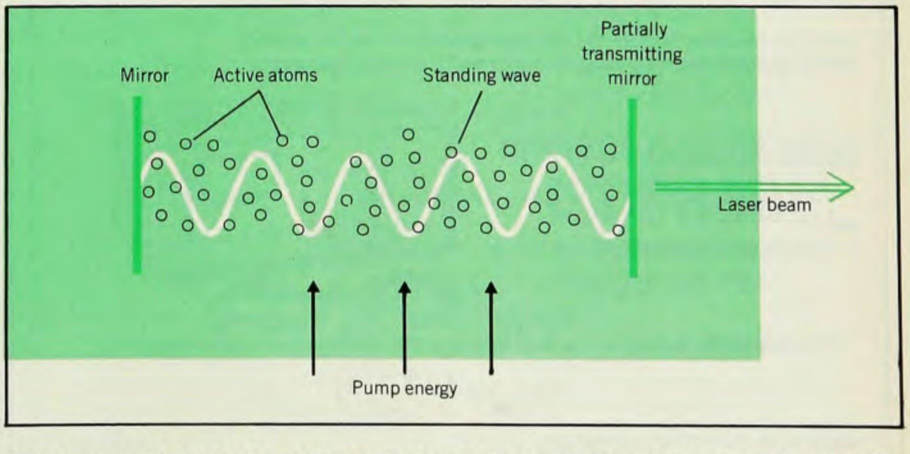
$$\frac{dE}{dt} = -\frac{dG}{dE}$$

gives the dynamical relation 1. The behavior of $G(E)$ is shown in figure 3. When we take into account the fact that stable points of the system correspond to minima in the potential, we immediately realize from this figure that below threshold the steady-state solution is $E = 0$, whereas above threshold such a solution is not a stable one.

Laser fluctuations

So far we have considered the laser field as a deterministic variable; however, fluctuations can not be neglected in the threshold region. For instance, it is clear from equation 1 that, without fluctuations, the laser above threshold can not evolve toward its steady state starting from the initial vacuum condition of zero laser electric field.

The most important source of noise in the laser is spontaneous emission from the upper to the lower laser level. A quantum theory that intrinsically accounts for fluctuations has been developed.[6,7] A first important result of the theory is the expression for the probability distribution $P(E)$ of the laser field near threshold.



The operation of a gas laser. The laser is an open system; at steady state, energy flows into the cavity (the pump energy from an external source) and an equal amount flows out of it (the coherent output beam). The laser system is not in thermal equilibrium, and the population of the energy levels of the active atoms in the cavity does not follow the Boltzmann distribution. Above the threshold for population inversion, the laser is in an ordered state, with the electromagnetic energy in the cavity in a single standing wave. The transition to this ordered state is parallel to such other transitions as the onset of ferromagnetism. Figure 1

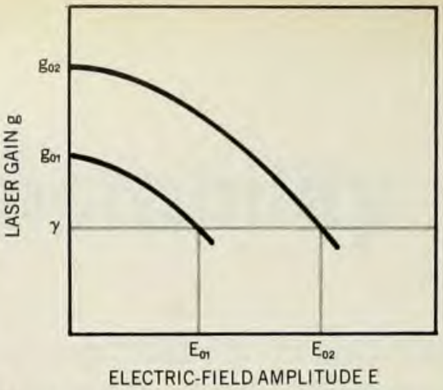
The function $P(E)$ can be put into the very suggestive form[1]

$$P(E) \propto \exp\left[-\frac{G(E)}{k\sigma}\right] \qquad (4)$$

where k is proportional to the spontaneous emission rate per atom. Note that $P(E)$ does not depend on the phase ϕ of the laser field.

Below threshold $P(E)$ is a bell-shaped function with a peak at $E = 0$. Well below threshold the quartic term in $G(E)$ can be neglected, so that $P(E)$ reduces to a gaussian function of E. Above threshold $P(E)$ is maximum for the value $E = E_0$, given by equation 2. It is then clear that the steady-state solution found when fluctuations are neglected gives the most probable value (not the mean value) of the laser electric field. The lowest-order moment of $P(E)$ having an operational meaning is the average intensity $I = \langle E^2 \rangle$. Very recent experimental results giving I as a function of the reduced normalized population inversion, $\epsilon = \sigma/\sigma_t - 1$, are shown in figure 4.

We can infer the magnitude of field-amplitude fluctuations qualitatively by considering the potential curves in figure 3 and using the fact that energy fluctuations due to spontaneous emission depend very weakly on $\sigma - \sigma_t$ in the threshold region, $\epsilon \ll 1$. For instance, it is immediately seen that the same change in the laser potential at the steady state produces a change in the field amplitude that is very large at threshold but relatively small above threshold, where the presence of a strong restoring force has a stabilizing effect on the field amplitude. More qualitatively, it can be shown that the mean-square fluctuation is proportional to $|\sigma - \sigma_t|^{-1}$.



The laser gain g as a function of the laser electric field E. Above threshold, g greater than γ, spontaneous emission noise is amplified, and the electromagnetic energy density in the cavity grows, decreasing the gain until it equals the loss γ. The figure shows two cases, characterized by unsaturated gains g_{01} and g_{02}, with corresponding steady-state amplitudes, E_{01} and E_{02}, of the laser field. Figure 2

Up to now our attention has been focussed on $P(E)$, in view of the analogy with the second-order phase transitions we shall discuss later. It should be recalled, however, that measurements of the statistical properties of laser beams give the photon count distribution $p(n)$, and not $P(E)$. The distribution $p(n)$ is directly related to $P(E)$: A gaussian probability density for the field (for a laser well below threshold) gives a Bose–Einstein photon distribution; a delta-function-like probability density (for a laser well above threshold) produces a Poisson distribution.[8] The gradual change in the photon probability distribution as the laser field goes from below to above threshold through a succession of stationary states has been measured[8] and compared successfully with theory.

The evolution of the statistical properties was also studied in a dynamical regime. A single-mode laser well below threshold is suddenly brought well above threshold by reducing the cavity losses through a fast electro-optical shutter.[10] The experimentally determined evolution of $p(n)$ from the unstable initial state to the final stationary state is shown in figure 5. The growth of the average photon number is exponential for short times; it is finally limited by saturation effects. Particularly interesting is the evolution of the variance of $p(n)$, which grows exponentially in the linear regime and goes through a maximum before attaining its steady-state value.

Turning now to the dynamics of laser-field fluctuations, we simply recall that the relaxation time of the amplitude fluctuations τ_c comes out to be inversely proportional to the curvature of the laser potential at the minimum, therefore varying as $|\sigma - \sigma_t|^{-1}$. The behavior of τ_c^{-1} is given in figure 6, which shows that very close to threshold the simple power-law dependence on $\sigma - \sigma_t$ breaks down, and a rounding-off effect appears. A similar fact can be observed in the plot of the average intensity I, figure 4. Both of these effects are due to the finiteness of the volume occupied by the laser mode; they are fully explained by the theory.

The phase-transition analogy

After this brief description of the instability and the fluctuations of the laser, let us pursue the analogy of the laser with second-order phase transitions. For simplicity we will refer to the ferromagnetic transition, but all our considerations are equally valid for such other second-order phase transitions as the liquid–vapor transition of a pure fluid near the critical point.

The Box on the left summarizes those properties of the ferromagnet near the Curie temperature T_C that are relevant to our discussion. From it and equation 2 we note that the dependence of the spontaneous magnetization M on the temperature T is formally identical to the dependence of the laser field E on the population inversion σ. The electric field

Critical properties of a ferromagnet

In the Landau theory the spontaneous magnetization M is the order parameter of the ferromagnetic transition, the magnetic field H is the variable conjugate to M and T_C is the Curie temperature.

The thermodynamic potential Φ is, near T_C,

$$\Phi(M,T,H) = \Phi_0 + \frac{a(T-T_C)}{2}M^2 + \frac{b}{4}M^4 - MH$$

which gives as the equation of state:

$$\frac{\partial\Phi}{\partial M} = a(T-T_C)M + bM^3 - H = 0$$

From this equation of state the following properties can be derived:

- the spontaneous magnetization for $H = 0$ (coexistence curve),

$$M \propto (T_C - T)^\beta, \text{ with } \beta = \frac{1}{2}$$

- the critical isotherm, $T = T_C$,

$$H \propto M^\delta, \text{ with } \delta = 3$$

and

- the magnetic susceptibility for $H = 0$,

$$\chi = \left(\frac{\partial M}{\partial H}\right)_{H\to 0} \propto |T - T_C|^{-\gamma}, \text{ with } \gamma = 1.$$

The probability density of the fluctuations of the spontaneous magnetization is

$$p(M) \propto \exp\left(-\frac{\Phi - \Phi_0}{k_B T}\right)$$

where k_B is the Boltzman constant.

The mean square of the fluctuation δM of the spontaneous magnetization is proportional to the susceptibility,

$$\langle \partial M^2 \rangle \propto \chi \propto |T - T_C|^{-\gamma}$$

near the Curie temperature.

E is therefore identified as the order parameter in the laser transition. Furthermore, because the steady-state value of E depends on $-(\sigma_t/\sigma - 1)$ in the same way that M depends on $T_c/T - 1$, σ corresponds, apart from the sign, to T.

The analogy is more than a formal one. Both the laser and the ferromagnet can be considered as positive-feedback systems, with loop gains less than one below threshold (above the critical point), and larger than one above threshold (below the critical point). The feedback acts on E in the laser and on M in the ferromagnet. The population inversion σ and the temperature T are the variables describing the interaction of the system with a reservoir—the pump for the laser and the thermostat for the magnet.

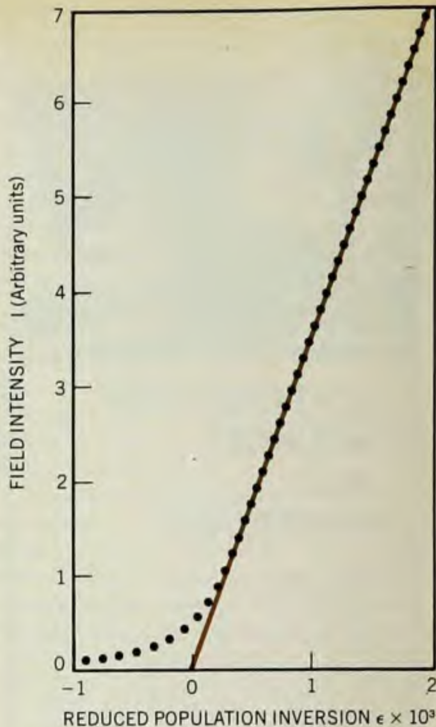
The change in sign that had to be introduced to make σ correspond to T is clearly due to the fact that the ordered state appears above σ_t for the laser but below T_c for the ferromagnet. Some critical systems thus have their ordered state above their critical temperature.

The analogy was further developed[1,11] by noting that the symmetry-breaking mechanism in the laser problem involves a classical signal injected into the laser cavity. This signal, resonant with laser field, generates an additional polarization S in the active medium. The quantity S is the variable conjugate to the order parameter E and therefore plays the same role as the external magnetic field in the ferromagnet.

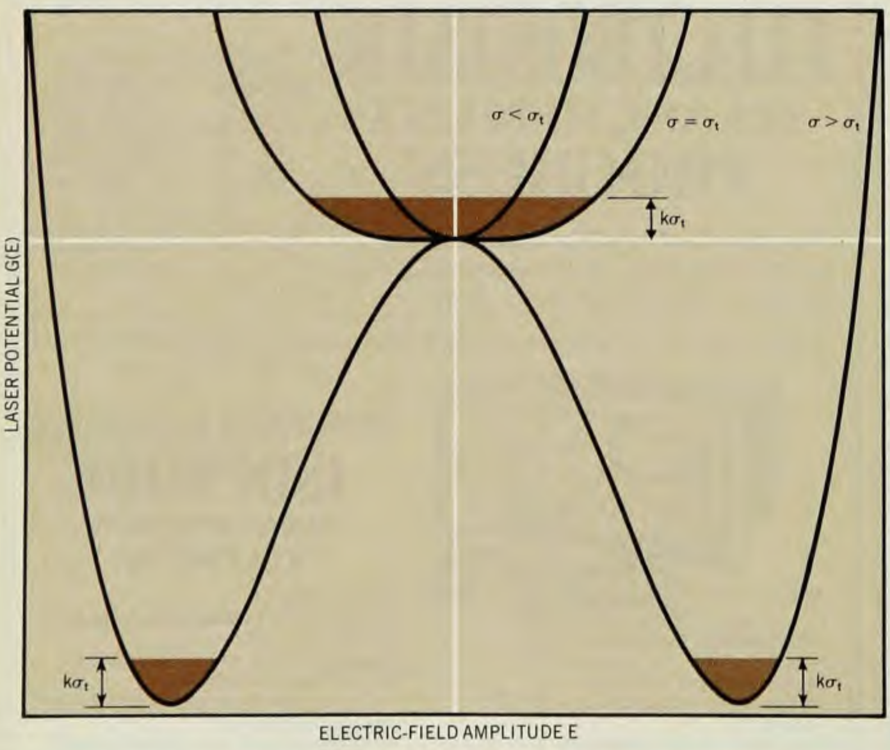
The introduction of S into the theory allows us to write an equation of state for the laser, which contains equation 2 as a particular case ($S = 0$). For $\sigma = \sigma_t$ the equation of state gives the critical inversion curve, $S \propto E^\delta$ with $\delta = 3$, as expected from the behavior of the critical isotherm (see the Box). Furthermore, the laser "susceptibility" $\chi = \partial E/\partial S$ can be derived from the equation of state. For $S = 0$, $\chi \propto |\sigma - \sigma_t|^{-\gamma}$ with $\gamma = 1$. The divergence of χ was recently measured[9] and found in agreement with the theoretical prediction.

If we compare equation 4 for $P(E)$ with the formula for $P(M)$ (in the Box on page 44), we find what is perhaps the most striking result of the analogy: The expression describing the probability density of field fluctuations has the same structure as the well known thermodynamic formula for the probability density of a thermodynamic variable. The function $G(E)$ plays the same role here as the free energy of a thermodynamic system, and the term $k\sigma$, which describes spontaneous emission noise, substitutes for the thermal noise energy $k_B T$.

At this point we can see that the description of the statics and dynamics of laser fluctuations can be rephrased entirely in terms of this analogy. I will only point out that the slowing down of laser-field fluctuations near σ_t can be related to the appearance of a "soft mode" in the second-order phase transition and that some of the features of the behavior of laser statistics during a transient (see figure 5) have also been found in a theoretical treatment of the evolution of a thermodynamically unstable system.[12] This last problem lately has received considerable attention in connection with the spinodal decomposition, which is the process of unmixing in an unstable solution.[13]



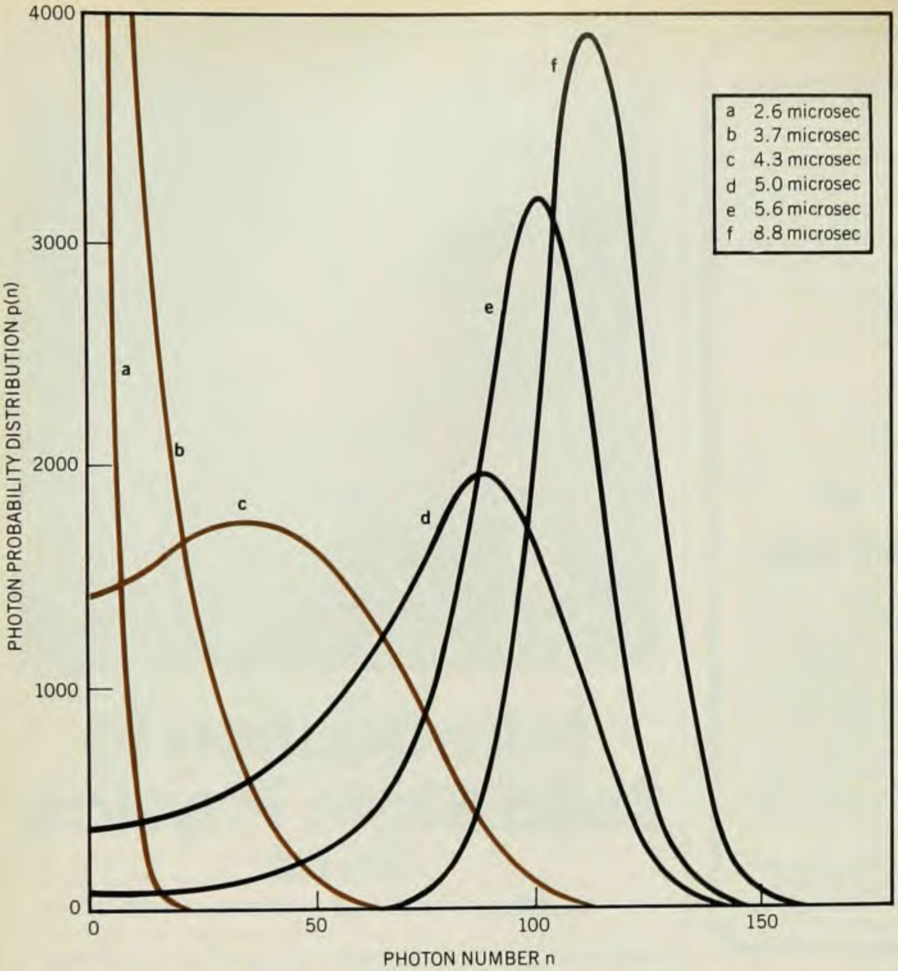
The average intensity I of the laser field as a function of the reduced population inversion $\epsilon = \sigma/\sigma_t - 1$. Above threshold, apart from a narrow region very close to the threshold point $\epsilon = 0$, I is proportional to ϵ. The experiment is described in reference 9. Figure 4



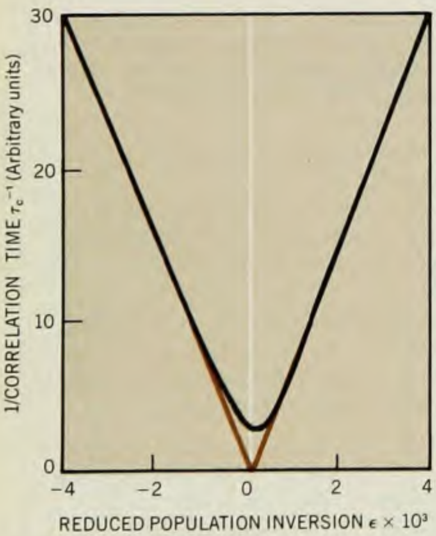
The laser potential $G(E)$, the minima of which give the steady states of the laser. The three cases shown are: below threshold (unsaturated population inversion σ less than its threshold value σ_t), at threshold and above it. The magnitudes of the fluctuations of the laser field E at steady state are found by considering energy fluctuations $k\sigma_t$ (see equation 4) above the minima of $G(E)$ and deriving the corresponding changes in the field amplitude. Figure 3

It should be stressed that the analogy discussed so far is between two theoretical models. It turns out that, whereas the laser model appears to be completely adequate to describe the behavior near threshold of actual laser devices, a serious disagreement with the Landau model was found in experiments on ferromagnets and other critical systems.[14] New approaches based on the scaling laws[15] and the renormalization-group theory[16] have been successfully proposed to interpret the experimental results. Although a thorough discussion of this point would lead us far beyond the scope of this article, we might recall here that a crucial parameter in the description of a cooperative system is the range of interaction of the ordering forces. In ferromagnets and fluids only nearest-neighbor interactions appear to be relevant; in the laser case, on the contrary, the range of interactions among the active atoms can be as large as the length of the optical cavity.

It is well known that an important parameter in critical phenomena is the correlation length of order-parameter fluctuations. Such a correlation length can



The evolution of the photon probability distribution $p(n)$ during a laser transient. The curves are for various time delays after switching, as given in the inset. The lines connect experimental points, which are not shown. The distributions, all normalized to the same area, change smoothly from the initial Bose–Einstein distribution **a**, typical of a chaotic source, to the final Poisson distribution **f**, typical of a coherent source. This distribution is related to the probability distribution $P(E)$ of the laser field. (From reference 10.) Figure 5



The reciprocal of the correlation time τ_c of laser-intensity fluctuations plotted versus ϵ. Apart from the threshold region ($\epsilon \simeq 0$), where rounding off occurs because of the finite laser volume, the correlation time follows a simple power law, which has an analog in the Landau ferromagnetism theory. The measurements are described in reference 18. Figure 6

be introduced also for the laser in a multimode theory.[2] There are several points not yet clarified in the multimode laser theory, however, and no experimental results near threshold are now available.

Hydrodynamic instabilities

As I indicated at the beginning of this article, there are many examples of open systems that undergo a transition from a disordered to an ordered state when the energy flux through the system exceeds a critical value. Let us briefly examine some instabilities in fluids[17] that have received much attention lately and that share many features in common with the laser instability.

We first consider the convective instability, also called the Rayleigh–Bénard instability. A fluid layer is confined between two horizontal plates at two different temperatures, with the lower-plate temperature T_1 larger than the upper-plate temperature T_2. As long as $\Delta T = T_1 - T_2$ is not too large the fluid remains quiescent and heat is transported by conduction. If ΔT exceeds a critical value ΔT_c, however, the fluid starts to move. Most surprisingly the convection pattern is very regular, taking the form either of rolls or of hexagons.

The physical origin of the instability lies in the competition between two effects:

- buoyancy, which drives the warmer, lighter fluid from the lower to the upper plate and vice versa, and
- heat diffusion, which locally equalizes the temperature.

If the heat diffusion is too fast, the fluid has no time to move. However, the buoyant force increases with ΔT, and so there will be a critical value of the temperature difference above which convection takes place.

An appropriate choice of the geometry of the experiment, shown in figure 7, gives a convection pattern consisting of straight, parallel rolls. Slightly above threshold the spatial dependence of the vertical and horizontal velocity components is sinusoidal along a suitable horizontal axis (the x axis in figure 7). The amplitude v_0 of this velocity oscillation plays the role of the order parameter in this case. A beautiful experiment performed by Pierre Bergé and Monique Dubois[16] with an optical heterodyne technique has shown that v_0 depends on $\Delta T - \Delta T_c$ as expected from a Landau-type description of the transition.

A theoretical treatment of the Rayleigh–Bénard instability that is quite parallel to the treatment of the laser instability was presented by Robert Graham[16] and others. They found, in particular, that the behavior of the fluid near the instability threshold is determined by a generalized potential of the same form as the thermodynamic free energy Φ in the Landau theory of phase transitions.

Some interesting effects can arise if the Bénard cell is filled with a binary liquid mixture (or a solution of macromolecules) instead of a pure fluid. It is known that a temperature gradient in the fluid layer causes a concentration gradient in the cell; this is known as the thermal-diffusion, or Soret, effect. Usually the heavier component migrates toward the colder plate (positive Soret coefficient), but in some cases it migrates toward the warmer. The existence of such a concentration gradient can alter substantially both the threshold value for convection of the temperature difference between the two plates, and the characteristics of the convective flow pattern above the threshold.[17] A recent experiment by Marzio Giglio and Antonio Vendramini clearly demonstrated the existence of an instability in an aqueous solution of polymers (heated from above!) having a negative Soret coefficient.

The last example we mention is the Taylor instability, which occurs in a fluid confined between an outer stationary cylinder and an inner rotating one. If the rotation frequency f of the inner cylinder is small, the motion of the fluid is purely azimuthal, with a local velocity that only depends on the distance from the axis of the two coaxial cylinders. However, if the rotation rate exceeds a critical value f_c, there arises, superimposed on the azimuthal flow, a toroidal roll pattern much like that of the Rayleigh–Bénard instability. The photograph in figure 8 shows this pattern, made visible with aluminum powder.

The spatial distribution of the velocity has been measured for the Taylor instability by Jerry Gollub and Michael Freilich[16] with the same optical technique as that used for the Rayleigh–Bénard instability. Here again the experimental results near the instability threshold were interpreted by introducing a generalized Landau potential. This example once more shows the parallelism between systems entering new ordered states.

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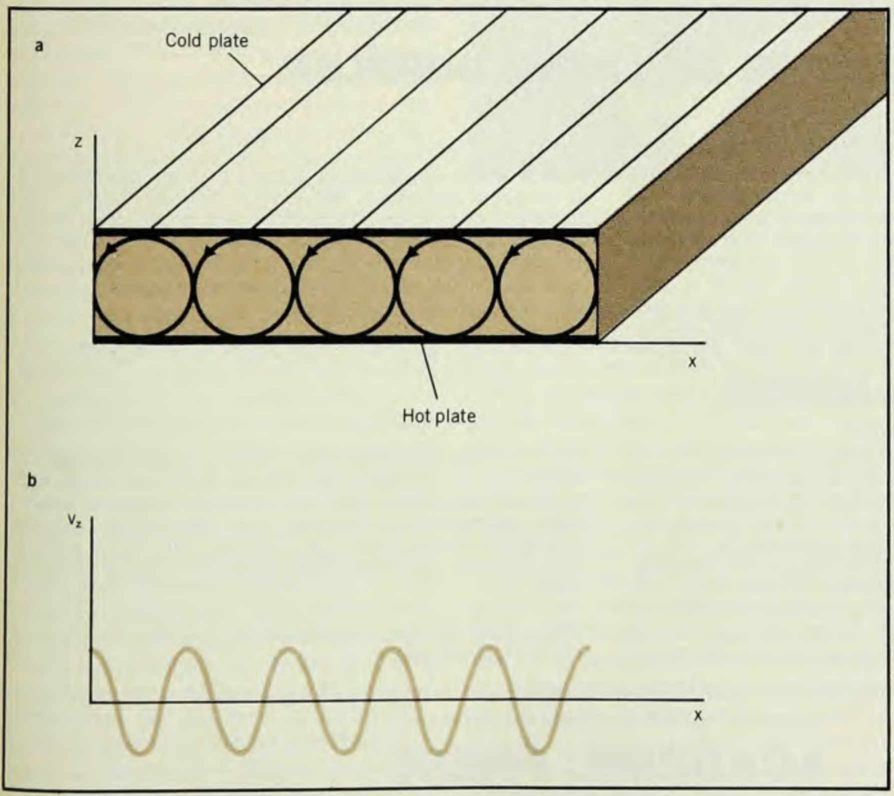
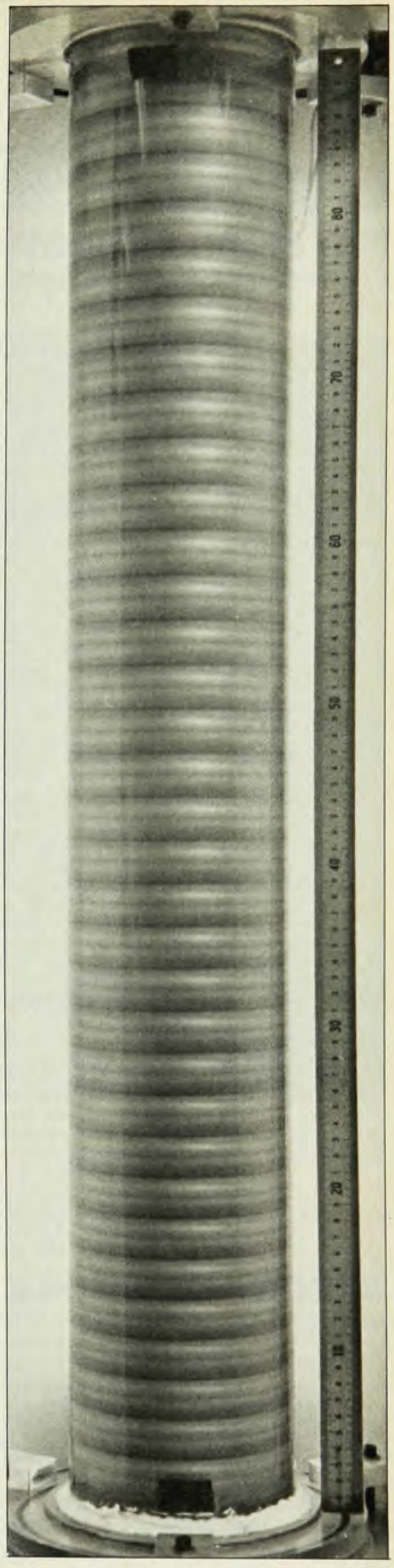


A rectangular Bénard cell exhibits a pattern of rolls, **a.** Slightly above threshold the velocity component v_z varies sinusoidally along the x axis, as shown in **b.** Figure 7



Taylor vortices in oil between a resting glass cylinder and an inner rotating cylinder. The fluid motion is made visible with aluminum powder; heavy lines show outward, thin dark lines inward, radial motion (Burkhalter, Koschmieder, J. Fluid Mech. **58**, 547; 1973). Figure 8