A theory of competitive running

Using simple dynamics one can correlate the physiological attributes of runners with world track records and determine the optimal race strategy.

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World records for running provide data of physiological significance. In this article, I shall provide a theory of running that is simple enough to be analyzed and yet allows one to determine certain physiological parameters from the records. The theory, which is based on Newton's second law and the calculus of variations, also provides an optimum strategy for running a race.

The records give the shortest time T in which a given distance D has been run. Our theory determines this function theoretically in terms of the following physiological quantities: the maximum force a runner can exert, the resistive force opposing the runner, the rate at which energy is supplied by the oxygen metabolism, and the initial amount of energy stored in the runner's body at the start of the race. By fitting the theoretical curve to four observed records, these quantities can be determined and the other records can be predicted. Alternatively, by measuring these quantities by independent physiological studies, one can use the theory to predict all the track records to which it applies.

The theory accounts for the main features of the records at distances from 50 meters to 10 000 meters. However, it does not account for the records at larger distances. These range up to 59 days for 5560 miles, the distance from Istanbul to Calcutta and back, set by M. Ernst (1799–1846).[1] Other physiological factors must be included to account for these records.

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Optimal running strategy

A runner's speed varies during a race; we assume that the speed, $v(t)$, is chosen in the way that minimizes the time T required to run the distance D, subject to physical and physiological limitations. We shall determine this optimal speed variation $v(t)$ by formulating and solving a mathematical problem in optimal control theory.

The theory predicts that the runner should run at maximum acceleration for all races at distances less than a critical distance $D_c = 291$ meters. Thus the races at distances less than 291 meters should be classified together as "short sprints" or "dashes." For D greater than 291 meters the theory predicts maximum acceleration for one or two seconds, then constant speed throughout the race until the final one or two seconds and finally a slight slowing down. This result confirms the accepted view that a runner should maintain constant speed to achieve the shortest time, and refines that view by fitting the constant speed to appropriate variable speeds during the initial and final seconds.

The mathematical solution for the optimal velocity

For $D \leq D_c$ the critical distance we have $f(t) = F$, and equations 2 and 3 yield

$$v(t) = F\tau(1 - e^{-t/\tau}) \qquad \text{(A1)}$$

Then equation 1 becomes

$$D = F\tau^2\left(\frac{T}{\tau} + e^{-T/\tau} - 1\right) \qquad \text{(A2)}$$

This gives the relation between D and T for $D \leq D_c$. To find $E(t)$ we use equation A1 for $v(t)$ in equations 5 and 6 to obtain

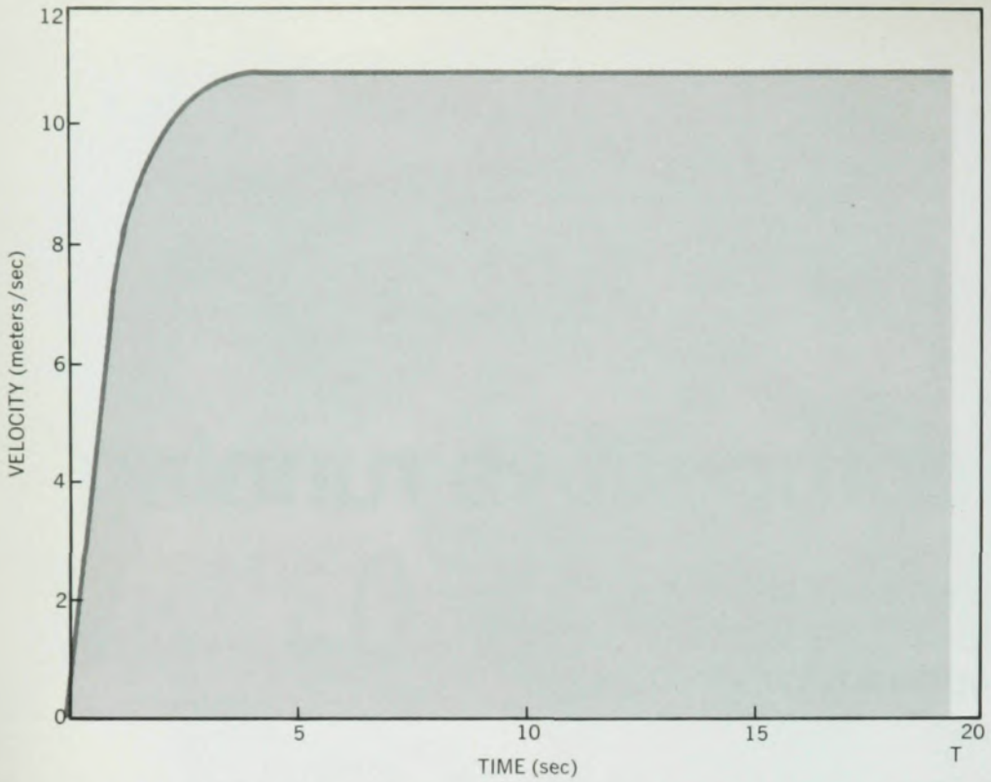
$$E(t) = E_0 + \sigma t - F^2\tau^2\left(\frac{t}{\tau} + e^{-t/\tau} - 1\right) \qquad \text{(A3)}$$

By setting $E(T_c) = 0$ we get the largest value of t for which the assumption $f = F$ is consistent with equation 7. Then D_c is the value of D given by equation A2 with $T = T_c$.

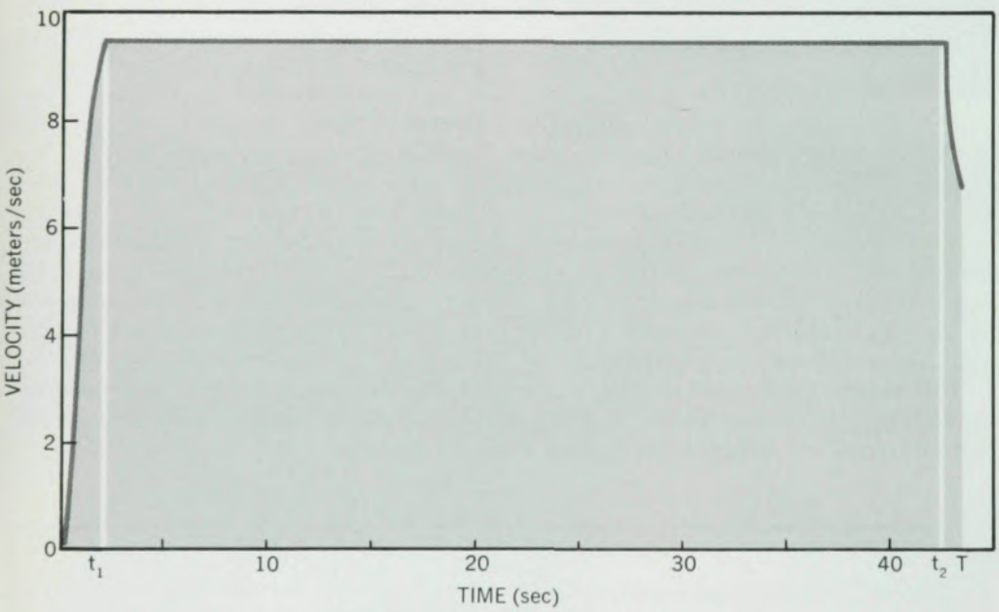
For $D \geq D_c$, $v(t)$ is given by A2 for $0 \leq t \leq t_1$, and by $v(t) = v(t_1) =$ constant for $t_1 \leq t \leq t_2$. In the interval $t_2 \leq t \leq T$, $v(t)$ is obtained by setting $E(t) = 0$ in equation 5, which yields $f = \sigma/v$. With this value of F, equation 2 can be solved with the result

$$v^2(t) = \sigma\tau + [v^2(t_1) - \sigma\tau]e^{-2(t_2 - t)/\tau} \qquad \text{(A4)}$$

The times t_1 and t_2 can be found by computing D from equation 1, using the three expressions for $v(t)$ just given, and then maximizing D with respect to t_1 and t_2. The fact that $v(t)$ equals a constant in the middle interval can be proved by the methods of the calculus of variations.



The 220-yard dash. The optimal velocity $v(t)$ is plotted versus t. The propulsive force $f(t) = F$ is maximum throughout the race. Figure 1



The 400-meter run. The optimal velocity $v(t)$ is plotted versus t. The propulsive force $f(t) = F$ is a maximum during the initial 1.78 seconds. After this initial acceleration, v remains constant until 0.86 seconds before the end of the race when the oxygen supply E becomes zero. Finally $E(t)$ remains zero during the last 0.86 seconds. This can be understood by thinking about a car with a limited amount of gas traveling over a distance D in the shortest time; the fuel should be used up shortly before the end. Figure 2

Runners at distances greater than 291 meters often finish with a kick rather than with the negative kick of the optimal solution. This discrepancy indicates either that they are not doing as well as they could or that the theory is inadequate. Presumably their goal is to beat competitors rather than to achieve the shortest time, and that goal influences their strategy. But if they ran at the optimal speed determined by the theory, they might do even better at beating competitors. Some trials in which runners attempt to follow the optimal strategy might determine which is the correct explanation, and whether the optimal solution is better than the usual strategy.

In our theory we assume that the resistance to running at speed v is proportional to v. Another assumption, suggested by measurements of R. Magaria and his collaborators,[2,3] is that the resistance is a constant independent of the velocity. We have worked out the theory in this second case for comparison and found that it leads to a quite unsatisfactory prediction, which indicates that it must be rejected if the other assumptions of the theory are correct.

Formulation of the theory

The length D of a race is related to the time T required to run it by the equation

$$D = \int_0^T v(t)dt \tag{1}$$

The velocity $v(t)$ is determined by the equation of motion, which we assume to be

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t) \tag{2}$$

In this equation $f(t)$ is the total propulsive force per unit mass exerted by the runner, part of which is used to overcome the internal and external resistive force v/τ per unit mass. It is an assumption that the resistance is a linear function of v and that the damping coefficient τ is a constant. Initially the runner is at rest; so

$$v(0) = 0 \tag{3}$$

The force $f(t)$ is under the control of the runner; so we may think of it as the control variable. The runner must adjust it so that T, determined by equation 1, is as small as possible when $v(t)$ is the solution of equations 2 and 3. There are two restrictions on $f(t)$. First, there is a constant maximum force per unit mass F that the runner can exert; so f must satisfy the inequality

$$f(t) \leq F \tag{4}$$

Second, the rate fv of doing work per unit mass must equal the rate at which the body supplies energy. This rate is limited by the availability of oxygen for the energy-releasing reactions, which we shall now consider.

Initially there is a certain quantity of available oxygen in the muscles, and more oxygen is provided by the respiratory and circulatory systems. It is convenient to measure the quantity of available oxygen in units of the energy it could release upon reacting. Thus we denote by $E(t)$ the energy equivalent of the available oxygen per unit mass at time t, by E_0 the initial amount, and by the constant σ the energy equivalent of the rate at which oxygen is supplied per unit mass in excess of the non-running metabolism. Then the equations of energy or oxygen balance can be written in the form

$$\frac{dE}{dt} = \sigma - fv \tag{5}$$

In addition E satisfies the initial condition

$$E(0) = E_0 \tag{6}$$

Because the energy equivalent of the available oxygen can never be negative, E also must satisfy the inequality

$$E(t) \geq 0 \tag{7}$$

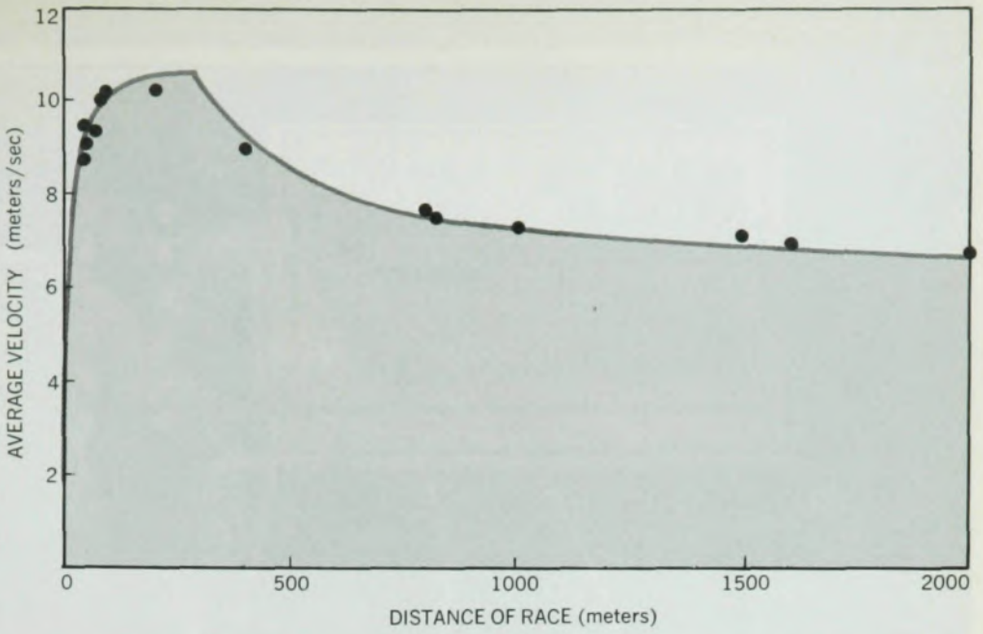
This is, indirectly, the second restriction of $f(t)$.

Now the runner's problem and ours is to find $v(t)$, $f(t)$ and $E(t)$ satisfying equations 2 through 7 so that T, defined by equation 1, is minimized. The four physiological constants τ, F, σ and E_0 are given, and so is the length of the race D. In other words, the problem is to find the rate of consumption of the initial oxygen supply in order to run the distance D in the shortest time.

My solution to this problem gives the result that for D not greater than D_c, where D_c is the critical distance mentioned above, $f(t) = F$, and v increases monotonically. For D greater than D_c, $v(t)$ increases for t less than t_1, v is constant for t between t_1 and t_2, and v decreases for t greater than t_2 until the end of the race, T. Figure 1 shows the optimal velocity $v(t)$ as a function of t for the 220-yard dash, which is a short sprint. In figure 2 the optimal $v(t)$ is shown for the 440-meter run, (D greater than D_c).

Comparison of theory and observation

A. V. Hill first pointed out the physiological significance of track records.[3] Since then, many investigators have tried to extract physiological information from these records,[1] but they were hampered by a lack of any theory of running that correlates the data. Twenty-two world records for distances from 50 yards to 10 000 meters are shown in Table 1. The first four are taken from data published by B. B. Lloyd,[4] and the others are from the *Reader's Digest Almanac 1972*, page 980. We have determined the two constants τ and F to yield a least-squares fit of the times given by the theory to the record times for the first eight races, which we assume to be short sprints. Then we determined σ and E_0 to give a least-squares fit of the times given by the theory for the remaining 14 races. In both cases we minimized the sum of the squares of the relative errors. The values of the four physiological constants obtained in this way, appear in Table 2. Also shown there is the value of D_c computed from the theory with these constants. We see that $D_c = 291$ meters is between 220 yards and 400 meters; so the first eight races are short sprints and the other 14 are not. The ratio of the initial oxygen supply, E_0, to the rate of oxygen supply σ is $E_0/\sigma = 58$ seconds. Thus the initial supply is equivalent to the oxygen that would be supplied by respiration and by circulation is 58 seconds.



The average velocity for running a race. The theoretical curve is seen to agree with average velocities calculated from world records. Figure 3

Table 1. Track Records

Distance D	Time T (record) min:sec	Time T (theory) min:sec	Error (per cent)	Average velocity D/T (theory) m/sec	t_1 sec	T-t_2 sec
50 yd	5.1	5.09	−0.2	8.99		
50 m	5.5	5.48	−0.4	9.12		
60 yd	5.9	5.93	0.5	9.26		
60 m	6.5	6.40	−1.5	9.38		
100 yd	9.1	9.29	2.1	9.85		
100 m	9.9	10.07	1.7	9.93		
200 m	19.5	19.25	−1.3	10.39		
220 yd	19.5	19.36	−0.7	10.39		
400 m	44.5	43.27	−2.8	9.24	1.78	.86
440 yd	44.9	43.62	−2.9	9.22	1.77	.86
800 m	1:44.3	1:45.95	1.6	7.55	1.07	1.08
880 yd	1:44.9	1:46.69	1.7	7.54	1.06	1.08
1000 m	2:16.2	2:18.16	1.4	7.24	0.98	1.16
1500 m	3:33.1	3:39.44	3.0	6.84	0.88	1.31
1 mile	3:51.1	3:57.28	2.7	6.78	0.87	1.34
2000 m	4:56.2	5:01.14	1.7	6.64	0.84	1.43
3000 m	7:39.6	7:44.96	1.2	6.45	.80	1.60
2 miles	8:19.8	8:20.82	0.2	6.43	.80	1.63
3 miles	12:50.4	12:44.89	−0.7	6.31	.77	1.80
5000 m	13:16.6	13:13.11	−0.4	6.30	.77	1.82
6 miles	26:47.0	25:57.62	−3.1	6.20	.75	2.10
10000 m	27:39.4	26:54.10	−2.7	6.20	.75	2.12

By using the constants in Table 2, we have computed the time given by the theory for each race. The results are shown in Table 1, together with the average velocity D/T given by the theory and the values of t_1 and $T - t_2$ for the races with D greater than D_c. The error in time between the theoretical value and the record is also shown in Table 1 as a percentage. We see that for the short sprints the error is at most 2.1%. However, for the longer races, it reaches 3.1% for the six-mile race. The average velocity given by the theory is plotted against distance in figure 3, which also shows the average velocities computed from the record times. Note that the initial increase and ultimate decrease of the average velocity is predicted by the theory quite satisfactorily. In comparing the theory with the actual records, it must be borne in mind that the theory

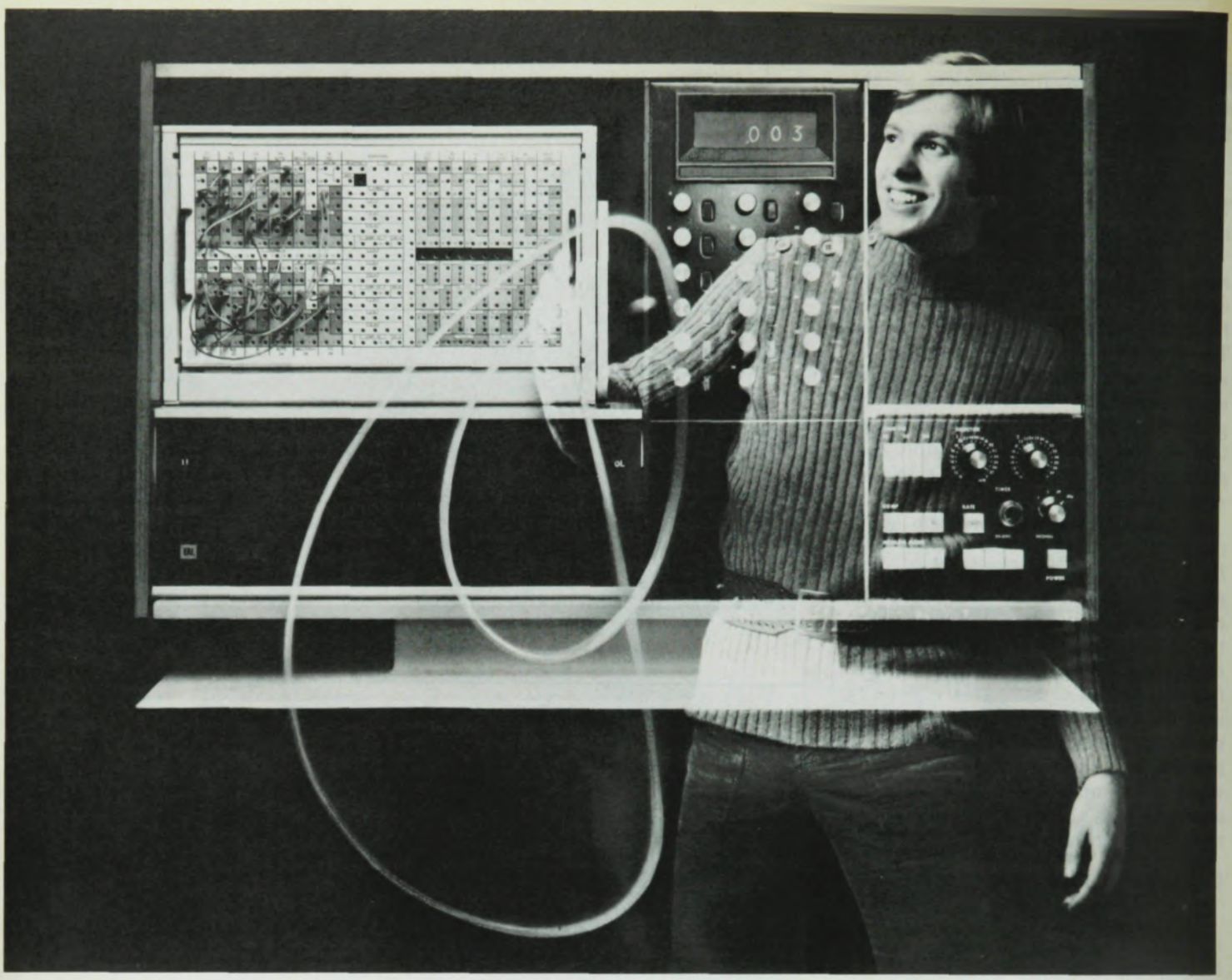


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Table 2. Physiological Constants

τ	= 0.892 sec
F	= 12.2 m/sec^2
σ	= 9.93 calories/kg sec
E_0	= 575 calories/kg
D_c	= 291 m

uses a single set of physiological constants at all distances, while the record holders at these distances undoubtedly had somewhat different constants from one another. The theory I have presented is too simple, because it omits various important mechanical and physiological effects. It ignores the up-and-down motion of the limbs; it fails to distinguish between internal and external resistance; it does not take into account the depletion of the fuel that uses the least oxygen and the transfer to the use of less efficient fuels; it ignores the accumulation of waste products and the mechanisms of removing them, and it probably ignores some other effects as well. A better theory incorporating some of these effects might be able to account for the records at longer distances, as well as those considered here. In support of such a new theory, measurements of the resistive force and the other physiological parameters are needed.

Nevertheless, this theory yields some definite results, and it would be of interest to adapt it to other types of races. Ice skating, swimming and bicycle races, for example, could be studied, taking into account the special features of each type of race. Another interesting problem would be to determine the influence of hills and valleys on the optimal velocity in longer races; this would require only a modification of the present theory.

* * *

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