The cosmic numbers

The large dimensionless numbers in cosmology have led to fascinating questions about the possible significance of their puzzling coincidences

Edward R. Harrison

Cosmic numbers have intrigued modern cosmologists ever since puzzling coincidences were first noticed between some of the large dimensionless constants. Here I will try to show not only that these numbers are interesting, but also that they are possibly important to our understanding of the physical world. Have the coincidences always existed or is it merely fortuitous that they occur at the present stage of evolution of the universe?

Discussions of the possible significance of cosmic numbers started with Hermann Weyl's paper[1] in 1919. Later, Sir Arthur Eddington,[2] Paul A. M. Dirac,[3] Pascual Jordan,[4] Robert H. Dicke,[5] Oskar Klein,[6] George Gamow[7] and many others found these numbers a source of inspiration in their studies of the universe. One of the most remarkable coincidences connected with cosmic numbers may be found in Archimedes' superb paper *The Sand Reckoner*[8] submitted to king Gelon, which anticipated by more than two thousand years the famous Eddington number of 10^{80} nucleons in the universe.

The ratio of the electromagnetic and gravitational coupling constants for an electron and a proton is the large dimensionless number N_1, defined as

$$N_1 = \frac{e^2}{Gm_n m_e} = 0.23 \times 10^{40} \qquad (1)$$

where e is the electron charge, m_n the nucleon mass, m_e the electron mass, and G the gravitational constant. The ratio of the strong[9] and gravitational coupling constants for two nucleons is also approximately N_1. This number plays a basic role in astrophysics. A spherical configuration of ionized hydrogen, in which long-range gravitational interactions are in equilibrium with short-range Coulomb interactions, contains approximately $N_1^{3/2}$ nucleons and has therefore a stellar mass of $m_n N_1^{3/2} = 90 M_\odot$ (where $M_\odot$ is the mass of the Sun).

Occasionally, in place of N_1 we find other numbers substituted; for example

$$N_1' = \frac{\hbar c}{Gm_n m_e} = 3.1 \times 10^{41} \qquad (2)$$

and

$$N_1'' = \hbar c / Gm_n^2 = 1.6 \times 10^{38}$$

where $\hbar$ is Planck's constant. Note that $m_n (N_1'')^{3/2} = 2M_\odot$.

The ratio of a characteristic cosmic distance and a characteristic intrinsic size of an elementary particle is

$$N_2 = \frac{L}{a} \approx 10^{40} \qquad (3)$$

where $L = c\tau$ is the Hubble distance, τ is the inverse of the Hubble parameter (or roughly the age of the universe,[10] about 10^{10} years), and a is either the classical electron radius, 2.8×10^{-13} cm, or the pion Compton wavelength, $\hbar/m_\pi c = 1.3 \times 10^{-13}$ cm.

Magic numbers

The order-of-magnitude coincidence

$$N_1 \approx N_2 \qquad (4)$$

between two dimensionless numbers as large as these is rather striking. This coincidence, and its possible deep physical significance, has provoked innumerable discussions and inspired a variety of cosmologies. Less well known, perhaps, is the sequence of "magic" or "cosmic" numbers

$$1, N^{1/2}, N, N^{3/2}, N^2, N^3, \ldots \qquad (5)$$

displayed in the Table (page 34). In this sequence, which exists because of the coincidence of N_1 and N_2, N denotes the $(N_1^p N_2^q)^{1/(p+q)}$ combination in which p and q are integers. The rough calculations in the box on page 34 indicate how these dimensionless numbers are obtained.

One can also interpolate, tentatively, quarter-integral powers of N. Photons of the 3K background radiation (and background neutrinos) are $N^{1/4}$ times as numerous as nucleons in the universe. This means that the entropy of the universe is $N^{9/4}k$, where k is Boltzmann's constant, and also the number of baryons and antibaryons is

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The sand reckoner

"There are some, king Gelon, who think that the number of the sand is infinite in multitude . . ."

Interest in large dimensionless numbers is not new. Archimedes may have been the first to wonder what was the largest physically significant number in the universe, but in his case he had to cope with what appears to us to be a difficult number system.

In the third century BC the Greeks had a numerical system with which they could count conveniently up to M^2, where M is a myriad, which is 10^4. Numbers greater than a myriad myriads in this system were expressed symbolically in a cumbersome and awkward fashion.[31] Archimedes introduced a numerical system,[32] "the naming of numbers," based on the Greek notation of counting from 1 to M^2, which was capable of expressing, concisely, extremely large numbers. In his system, the number expressed as "*p* units of the *q*th order and the *r*th period," is

$$pM^{2[(q-1)\ +\ (r-1)M^2]}$$

where *p*, *q* and *r* are integers in the range 1 to M^2.

In his paper, *The Sand Reckoner*,[8] Archimedes applied this numerical system to the problem of calculating the maximum number of grains of sand the universe could contain. With data from the work of Aristarchus he estimated that the diameter of "the sphere of fixed stars" in stadia is no more than a hundred myriad units of the second order and first period, and therefore the maximum number of grains of sand that could be contained within the sphere of fixed stars is a thousand myriad units of the eighth order and first period. Hence, in our notation, the maximum number of grains of sand is

$$10^7 \times 10^{8(8-1)} = 10^{63}$$

Archimedes' grains of sand were extremely small. He assumed that a poppy-seed has a diameter 1/10 of a finger-breadth and a volume equivalent to that of a myriad grains of sand. Taking a finger-breadth as 1 cm, and the density of sand grains as roughly 3 g cm^{-3}, we find that each of his grains of sand has a mass 1.6×10^{-7} g, and consists therefore of 1×10^{17} nucleons. Hence, according to Archimedes' calculations, the universe contains a maximum of $10^{63+17} = 10^{80}$ nucleons.

$N^{1/4}$ times the baryon number in the early universe.[14] The mass of a galaxy such as our own is $N^{7/4}$ nucleon masses, and is $N^{-1/4}$ the mass of the Hubble sphere. I find these numbers amusing, and it is clearly a matter of personal taste whether or not one attaches any real significance to the existence of these quarter-integral powers of N.

Dirac's principle

Dirac[3] has postulated: "*Any two of the very large dimensionless numbers occurring in Nature are connected by a simple mathematical relation, in which the coefficients are of order-of-magnitude unity.*" But N_2 is proportional to τ according to equation 3, and therefore those numbers compounded from N_2 vary with time as the universe expands and their present orderly relation breaks down at other epochs. And there is the rub. Is Dirac's principle and the pleasing sequence of integral, half-integral and perhaps quarter-integral powers of N the result of a mere fortuitous accident of our present era?

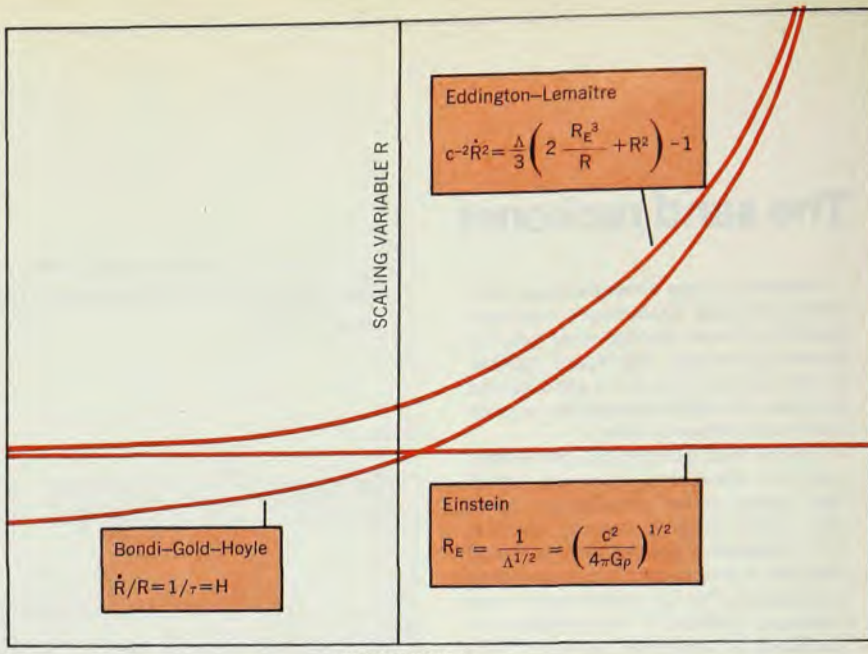
The cosmic-number dilemma posed by Dirac's principle and the sequence 5 can be side-stepped in various ways. In the steady-state theory[15] the Hubble distance, and therefore N_2, is forever constant. (See figure 1.) In the Eddington picture,[2] the cosmological constant Λ (or rather $\Lambda^{-1/2}$) and not the Hubble distance is the cosmic yardstick against which everything is measured; therefore N_2 remains permanently equal to N_1. Both are antibigbang models of the universe and therefore encounter difficulty in explaining the 3K microwave background radiation, which is thought to be a remnant of the big bang.[16]

Dicke[17] has proposed a remarkable solution of the problem. In the distant past N_2 was small when nobody was around to take note of its value. Intelligent life does not emerge until the first generation of stars evolve and produce by nucleosynthesis the requisite elements of complex biological systems. But by this time N_2 has attained a value similar to N_1 and no gross discrepancy is apparent. The lifetime of a star is typically

$$t_{\mathrm{star}} \approx 10^{-2}c^2\ \mathrm{mass/luminosity} \quad (6)$$

where 10^{-2} is approximately the fractional mass difference of a helium nucleus and four protons. The maximum possible luminosity–mass ratio $3Gm_n c/2a^2$ occurs when radiation pressure on the electrons in a stellar atmosphere equals the gravitational pull on the ions. Such a luminosity–mass ratio is excessive, and 1% of this value is more realistic for massive stars; therefore from equation 6 we obtain

$$t_{\mathrm{star}} \approx a^2c/Gm_n$$



Models of the universe that satisfy Dirac's principle. Einstein's original model is static, and therefore $N_2 = 1/\Lambda^{1/2}a$ is constant and equal to one of the N_1 numbers. The Einstein universe, however, is unstable and evolves into either a collapsing or expanding universe. The expanding universe in this case is represented by the Eddington–Lemaitre model, and the cosmic yardstick for determining N_2 is again $\Lambda^{-1/2}$. In the steady-state model of Bondi, Gold and Hoyle the Hubble distance is τc; then the cosmic number N_2, here equal to $\tau c/a$, is again constant, as it is in each of these three models. Figure 1

After the first generation of stars have burned their hydrogen and produced heavy elements the universe also has an age of order t_{star}. Whereupon, from equation 3

$$N_2 \approx \frac{ct_{\mathrm{star}}}{a} \approx \frac{e^2}{Gm_n m_e} = N_1 \qquad (7)$$

in accordance with Dirac's principle. This ingenious explanation leaves me, at least, with a vague and uneasy feeling that possibly some unknown fundamental relation still lurks between N_1 and N_2.

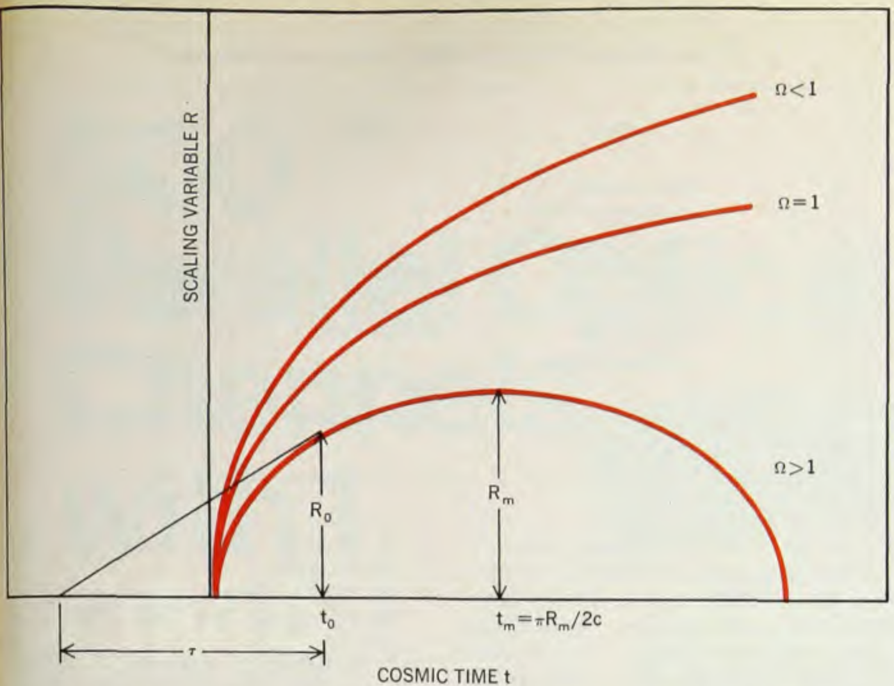
Lured like seafarers of old into hazardous waters by enticing sirens, the more courageous explore the possibility that the fundamental constants of nature are time-varying. If N_2 changes with time, perhaps N_1 changes with time also, in such a way as to preserve the *status quo?*

From terrestrial isotope abundances it is known[18] that variation in e, affecting Coulomb repulsion in the nucleus and beta-decay, has been extremely small during the Earth's history. Further, Gamow's suggestion[19] that e^2/hc changes significantly with time is ruled out by observations[20] of fine-structure splitting of emission lines in the spectra of quasars and distant galaxies. Terrestrial and astronomical consequences of time-varying G are complex, and it is difficult, except in the case of the Sun, to draw firm conclusions from the data available to us.[21]

If G varies in the following manner

$$G \propto \tau^{-n} \qquad (8)$$

we require n equal to 1 according to Dirac's principle. The Sun's luminosity is roughly proportional to G to the eighth power,[22] and therefore in the past the Sun consumed its hydrogen at a much higher rate than at present. (In effect, the Sun contained a number of nucleons larger than $N^{3/2}$ and hence ranked among massive stars that evolve rapidly.) E. Pochoda and Martin Schwarzschild[23] show that if n equals 1 the Sun evolves into a red giant in 1.5×10^9 years, which is only one third the age of Earth. A weaker variation, $n = 0.2$, is not ruled out, although the enhanced neutrino flux widens the discordance between theory and experiment.[24] The possibility that either G or particle masses are time-varying sweeps us willy-nilly into the deep waters of Mach's principle.[25] This principle, interpreted in various ways,[26] asserts that local inertial properties are governed by the global distribution of mass in the universe. So far, this bootstrap concept has not been successfully incorporated into the equations of physics. Scalar–tensor theories of the kind investigated by Jordan,[4] and Carl Brans and Dicke,[27] can be adjusted to give a G variation weak enough not to conflict with current observational data.[28] But weak variations equivalent to n not greater than 0.2, while conforming to the spirit



Friedmann models of the universe. In these models the cosmological constant Λ is zero. The universe commences as a "big bang" of high density and is of finite age. The universe is spatially finite and will eventually collapse when the density parameter Ω is greater than unity. The universe is spatially infinite when Ω is less than or equal to unity. The $\Omega = 1$ curve is the Einstein–de Sitter model. The Hubble time is τ and the characteristic yardstick $c\tau$ varies in time for all three models. Here R_0 represents the present radius of the closed model universe ($\Omega > 1$); R_m is the maximum value of the radius. Figure 2

of Mach's principle, fail completely to satisfy Dirac's principle.

New points of view

Ralph Alpher and Gamow have proposed[29] that N_2 is equal to t_{eq}/a, where $t_{eq} \approx 10^6$ years is the epoch in the early universe when radiation and matter densities are equal. Thus, $N_2 \approx 10^{36}$ in this case has a constant value, of the same order as $N_1 = e^2/Gm_n{}^2 = 1.2 \times 10^{36}$. By relating the cosmic numbers in this way, they are able to include the background radiation and derive a value for the entropy per nucleon that is in reasonable agreement with results from experimental observation.

A different approach, which so far as I know has not been previously suggested, resolves the cosmic-number problem in the following simple way. The present average density of the universe is

$$\rho_0 = 4.7 \times 10^{-30}(H_0/50)^2\Omega \text{ g cm}^{-3} \qquad (9)$$

where H_0 is the present value of the Hubble parameter measured in km sec^{-1} megaparsec^{-1} (note $H_0 = 75$ is equivalent to a recession of 1 cm sec^{-1} per light year). The density parameter Ω is twice the deceleration parameter for the Friedmann models (see figure 2) in which the cosmological constant Λ is zero. The universe is open and infinite and forever expands when $\Omega \leq 1$, and is closed and finite and must eventually collapse when $\Omega > 1$. If the universe is closed (which I shall assume) the Friedmann equation[10] is

$$\left(\frac{dR}{dt}\right)^2 = c^2\left(\frac{R_m}{R} - 1\right) \qquad (10)$$

having the parametric solutions

$$R = R_m \sin^2\psi \qquad (11)$$

and

$$t = R_m c^{-1}(\psi - \sin\psi\cos\psi) \qquad (12)$$

The maximum radius R_m is attained when the universe has the age $\pi R_m/2c$.

A closed universe possesses the attractive property that R_m may be used as a time-invariant cosmic yardstick for determining N_2, thus allowing us to write

$$N_2 = R_m/a \qquad (13)$$

in place of equation 3. In support of this idea we can argue that observations generally fail to provide, directly, a truly global picture of the universe. The universe is a manifold of both space and time, and therefore N_2, if it is of fundamental importance, should relate to the whole of space–time and not to measurements made on transitory spatial hypersurfaces of space–time.

In our closed model, R is the radius of spherical space of maximum radius

$$R_m = (3c^2/8\pi G\rho_m)^{1/2} \qquad (14)$$

at density ρ_m, of spatial volume $V_3 = 2\pi^2R^3$ and total space–time volume

$$V_4 = 2\pi^2\int_{\psi=0}^{\psi=\pi} R^3 dt = \frac{35\pi^3 R_m{}^4}{32c}$$

Hence, the total four-volume of the universe is 10^{160} f^{3}j (f = fermi = 10^{-13} cm, j = jiffy; *the jiffy unit is due to* R. C. Tolman and is the time taken by light to travel 10^{-13} cm), and each of the 10^{80} nucleons is allotted a four-volume of 10^{80} f^{3}j and occupies 10^{40} f^{3}j. From equations 11, 12 and 14 we find the present radius is

$$R_0 = R_m\left(1 - \frac{1}{\Omega}\right) \qquad (15)$$

and since $H_0 = (dR/dt)_0/R_0$, it follows that

$$R_m = \frac{c\Omega}{H_0(\Omega - 1)^{3/2}} \qquad (16)$$

and

$$\rho_m = \frac{3H_0{}^2(\Omega - 1)^3}{8\pi G\Omega^2} \qquad (17)$$

A principal objective in observational cosmology is to determine the parameters H_0 and Ω, and if possible find a Friedmann trajectory for the universe.[30] After these parameters have been determined, theoretical cosmology will then be confronted with the awesome task of explaining why the parameters have the observed values. This will mean, if the universe is closed, explaining the values of R_m and ρ_m. One cannot resist conjecturing that in this case the parameters R_m and ρ_m are related in some obscure way, directly or indirectly, with the cosmic numbers. As an example, if $H_0 = 50$ km sec^{-1} megaparsec^{-1}, and N_2 is given by equation 13 and is equal to N_1' of equation 2, we find $\Omega = 2$ and the universe is now half its maximum size.

Archimedes' number

Large dimensionless numbers are not entirely new in science, nor are they peculiar only to modern cosmology. Archimedes, in the third century BC, set himself the task of computing the largest number that he believed would have physical significance; this he expressed as the number of grains of sand that could be contained in the whole universe. The principles of his calculation, as presented to king Gelon, are summarized in the box on page 31. His result is the number 10^{63}. From his data on the assumed size of a grain of sand we can calculate that there are 10^{17} nucleons in each grain, which tells us that Archimedes' universe contains a maximum of 10^{80} nucleons.

Archimedes' estimate of the maximum number of grains of sand the universe could contain differs vastly from any modern estimate; furthermore, he naturally knew nothing about nucleons. Even when all this is taken into account, we can still marvel. The essential equivalence of Archimedes' number and Eddington's number must surely rank as one of the most amazing coincidences of the many we have met in

Magic numbers

1

The gravitational potential energy GMm_n/L of the Hubble sphere (that is, the observable universe of mass $M \approx \rho L^3 \approx L^3/G\tau^2 \approx Lc^2/G$, density ρ) and a nucleon, divided by the nucleon energy $m_n c^2$, is of order unity. A photon mean free path in a universe of uniformly distributed ionized hydrogen is $N_1^{-1} N_2$ times the Hubble distance L, with a^2 as the Thomson cross section.

$N^{1/2}$

The classical electron radius a is $N_1^{1/2}$ multiplied by the Planck[11] length $(G\hbar/c^2)^{1/2} = 1.6 \times 10^{-23}$ cm; the Planck mass $(\hbar c/G)^{1/2} = 2.1 \times 10^{-5}$g is $N_1^{1/2}$ nucleon masses; the radius of a neutron star or black hole of stellar mass is $N_1^{1/2}$ times a. The Hubble distance is $N_1^{-1/2} N_2$ times the radius of a neutron star; the mass of the Hubble sphere is $N_1^{-1/2} N_2$ stellar masses.

N

In addition to equations 1 and 3, we have that a is N_1 times the nucleon gravitational radius Gm_n/c^2; the hadron barrier[12] density c^2/Ga^2 (when the Hubble distance equals a) is N_1 times the nuclear density m_n/a^3. Nuclear density is $N_1^{-1} N_2^2$ greater than the mean density c^2/GL^2 of the universe; the ratio[13] of gravitational and electromagnetic forces between two hydrogen atoms separated a distance L is $N_1^{-1} N_2^2$.

$N^{3/2}$

A stellar mass equals $N_1^{3/2}$ nucleon masses. The Hubble distance is $N_1^{1/2} N_2$ Planck-length units; the mass of the Hubble sphere is $N_1^{1/2} N_2$ Planck-mass units.

N^2

Mass of the Hubble sphere divided by the nucleon mass is the famous Eddington number of $N_1 N_2 \approx 10^{80}$ nucleons in the universe.[2] The Planck density $c^5/G^2\hbar$ = (Planck mass)/(Planck length)^3 is N_1^2 times the nucleon density.

N^3

Other dimensionless numbers can be found, such as $N_1 N_2^2$ for the ratio of the Planck density and the mean density of the universe.

Sizes, masses and densities scaled in terms of N

	Size	Mass	Density
N^{-1}	Gravitational radius of nucleus		Universe
$N^{-1/2}$	Planck length		
1	Classical electron radius	Nucleon	Nuclear density Neutron star
$N^{1/2}$	Neutron star	Planck mass	
N	Hubble distance		Hadron barrier
$N^{3/2}$		Star	
N^2		Hubble sphere	Planck density

the whole domain of cosmic numbers.

This leads us to the sobering thought that if by sheer chance, with little probability of success, Archimedes could indirectly obtain the number 10^{80}, might not N_1 and N_2, improbable though it may seem, also have accidental coincidence? At present we do not know how to bootstrap together the microscopic and macroscopic realms of physics in order that N_1 and N_2 are necessarily similar numbers. Each of us according to his own inclination is therefore free either to dismiss the coincidence as fortuitous, or to think that the coincidence is evidence of an underlying grand design in the structure of the physical world.

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