A resolution of the clock paradox

It's no paradox at all, claims our author, to an observer who keeps in mind the meaning of space and time in relativity theory.

Mendel Sachs

Ever since the initial successes of the theory of relativity, physicists and philosophers of science have written a great deal about a paradox that seems to arise when theory tries to answer the question: If two identical stationary clocks in the same inertial frame of reference are synchronized, and if one of them is accelerated away into different inertial frames and then returned to the original inertial frame, would the time readings of the clocks still be synchronized at this later rendezvous?

The majority of authors,[1,2] including Einstein himself,[3,4] have argued, as we shall see, that relativity theory implies the answer to be no, the clocks would no longer be synchronized. My own resolution of the "clock paradox," which depends on general-relativity arguments, comes to the opposite conclusion: There would be no real loss of synchronization, even though there would be an apparent loss while the clocks are in relative motion.

Which twin is older?

The prediction of those who agree with Einstein has been that the clock that was accelerated away from the "fixed" clock, and then brought back to it, would read slower than the clock that did not have this history of travel. This conclusion has an interesting implication: If an identical twin parts company with his brother by taking a trip into outer space in a rocket ship at sufficiently great speed, then when he returns to Earth he should be noticeably younger than his brother. Indeed, for sufficient speeds, he may even be able to return to his own great grandchildren, who may be older than he is!

This conclusion depends on a tacit assumption, made originally by Einstein, that there must be a one-to-one correspondence between an observer's time measure on a clock that moves relative to him, and a physical process, such as aging, that occurs in that moving system. (For a clock, aging would be the unwinding of a spring.) Of couse we can always fix our own aging process as a standard from which we can calibrate the clocks in our own frame of reference, or vice versa. But this correlation between a fixed observer's time measure on a moving clock and its aging process is not necessarily true. Nevertheless, should one postulate such a correlation, then "asymmetric aging" does seem to occur.

On the other hand, a different interpretation appears to me to follow logically from the meaning of "time" in relativity theory and leads to a conclusion that necessarily rejects this correlation between relative time measure and aging. A fundamental distinction is made in relativity theory between the space–time parameters on the one hand and, on the other hand, the physical interactions themselves, such as the aging process of a physical system. The space–time parameters are a useful language to *express* the laws of nature, whereas the physical interactions *obey* the laws of nature. Henri Bergson, who rejected[5] Einstein's contention, argued that the conclusion of asymmetric aging is perhaps closer to Lorentz's interpretation of the time dilation as a physical manifestation of an *absolute* ether (which he be-

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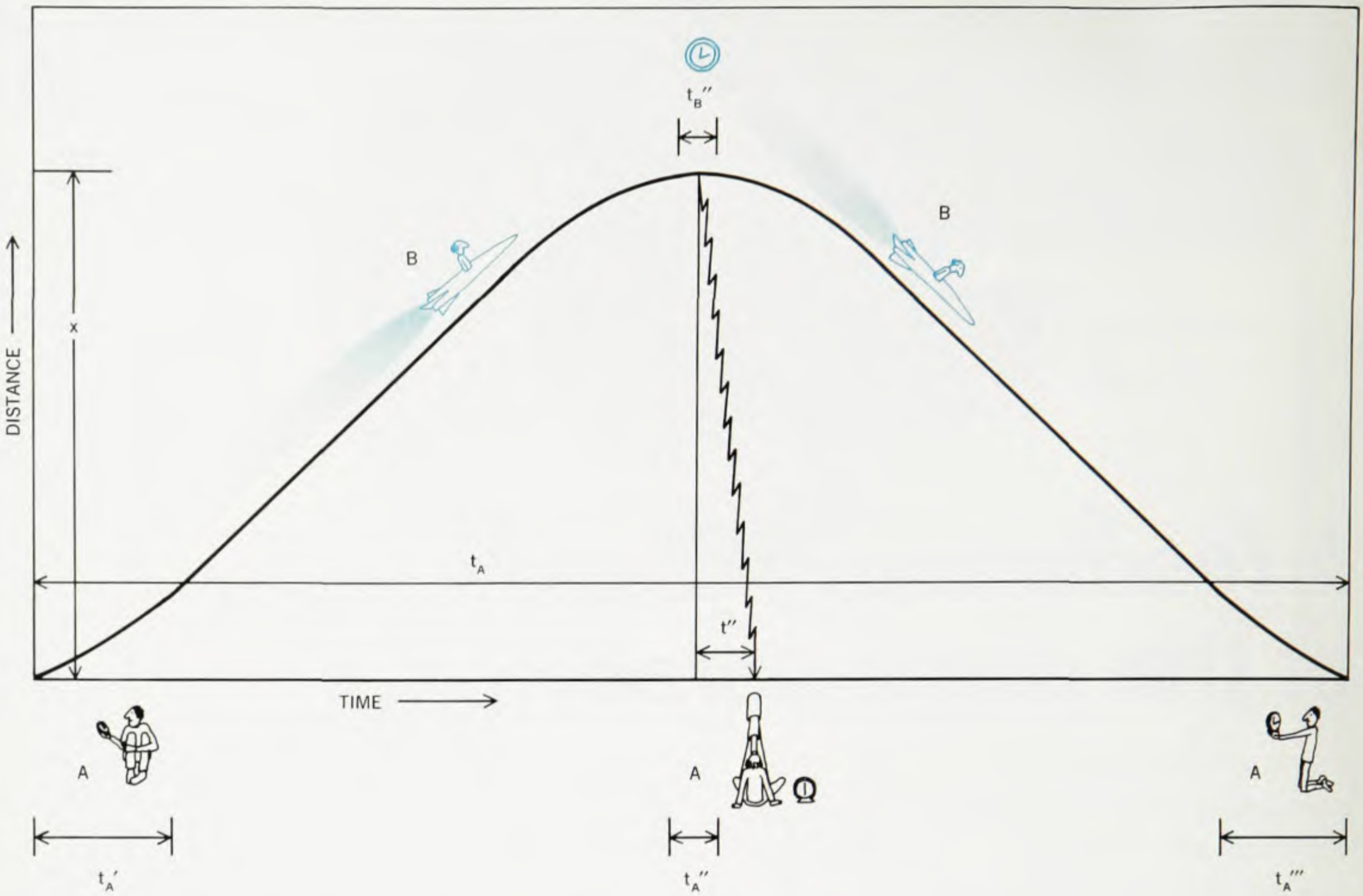

lieved to conduct light) than it is to Einstein's interpretation in terms of a *relative* measure.

Fixed versus moving frames

Before presenting these different views of the clock problem, I point out that this paradox has not yet been settled unambiguously, either by direct experimental observation or in an exact theoretical proof. Perhaps this is why the literature that attempts to settle the question is so voluminous. Many experiments have confirmed the time-dilation effect of special relativity: The time measure of a moving clock is different from a corresponding time measure of a fixed clock *from the view of the fixed frame.* But the clock problem does not deal only with this situation. It deals with two entities, say twin brothers, who are comparing notes while they are in the same frame of reference. One of them has had the experience of leaving Earth at some earlier time and then returning to compare the aging of his body with that of his brother, knowing that they were born of the same mother at the same time. The majority *opinion* today is that one of these twins would now be wrinkled, white haired, bent and weak limbed, whereas his traveling brother is still smooth skinned, brown haired, of straight posture and strong limbed.

Of course we have no experimental confirmation of this effect for people, because we have not yet been able to develop vehicles able to travel at speeds that approach the speed of light. But a confirmation of the effect needn't use people. It might depend on experimental observations of two equally massive pieces of radioactively decaying metal, cut from the same block. One of these pieces might be accelerated away and then brought back to the frame of reference of the piece that did not travel. After the two pieces of metal are again in the same inertial frame of reference, would one of them now reveal that fewer of its constituent nuclei had decayed than the nuclei of the other piece of metal? I am unaware of any direct experimental evidence to answer this question.

None of the theoretical conclusions thus far are based on explicit mathematical application of relativity theory to the particular situation considered, although numerous demonstrations have been based on mathematical approximations.[1,6] Einstein's suggested theoretical approach,[2] which uses the equivalence principle together with the transformations of special-relativity theory, has been exploited by many authors, most of whom agree with Einstein's conclusions that the asymmetric aging effect occurs. To some of us, however, such an approach is questionable, because the effect investigated depends on one part of a system making changes (by whatever means it chooses) from the inertial frame of the other part. That is, the transformations must describe not only uniform relative motion but also the *change* from one inertial frame to another, so that we must use general-relativity theory rather than an improvisation of the formalism of special-relativity theory if we wish to draw unambiguous conclusions.

Einstein[4] and Hans Reichenbach[2] attribute the physical effect—that the clocks have lost their original synchronization—to the gravitational action of the rest of the universe on the clock that is accelerated away from the stationary one (say on Earth). They contend that the clocks are not equivalent, a requirement if we are not to have a logical paradox, because from the frame of reference of the stationary clock, the second clock moves with respect to it and a stationary universe. On the other hand, from the frame of reference of the moving clock, the stationary clock and the mass distribution of the universe, relative to the earth, move away from it (in the opposite direction). In the first view then the universe is stationary, whereas in the second it is moving, making the two clocks inequivalent and thereby accounting for the alleged effect. Further on, we shall derive the effect, according to this view.

In the same article in which Einstein

Does asymmetric aging occur when one man (say twin B) accelerates swiftly into space, covering a distince 2x and returning to his brother (twin A) in a time t_A? According to Einstein, asymmetry occurs only during the period when B reverses for the return journey: A measures this turn-around time as t_A'' on his own clock and as t_B'' on B's clock; t'' is the time needed for the signal to travel between the two clocks. The acceleration and deceleration periods t_A' and t_A''' play no role, he points out, because A and B are at nearly equivalent positions in the universe at these times.

defends the asymmetric aging effect, he also asserts:

> "according to general-relativity theory, the four coordinates of the space-time continuum can be selected quite arbitrarily; for each is a parameter lacking independent physical significance."[4]

It appears to me, then, that his conclusion about the asymmetric aging effect is logically incompatible with his assertion about the significance of space-time in relativity theory. (Reference 7 includes an elaboration of this view of space and time.) My own acceptance of this interpretation of space-time is, as we shall see, the main basis for my logical rejection of his conclusion about the aging effect.

Asymmetric aging—pro

If we accept the idea of a linear relation between a fixed observer's time measure on a moving clock and the aging in the moving system, it follows that the aging process takes place at a slower rate in the moving system. This conclusion comes from the Lorentz transformation relating the respective time measure, δt_0 and δt, made by the fixed observer in the moving and in the fixed frame of reference

$$\delta t_0 = \delta t(1 - (v/c)^2)^{1/2} \qquad (1)$$

The logical paradox enters when we use the implication of relativity theory that there is no absolute frame of reference, so that all motion is relative with respect to the description of physical effects. Thus, the observer (A) who was previously called "fixed" and who stood looking at a clock that moves away from him at the speed v could equally be considered to be moving at the speed $-v$ with respect to a "fixed" observer (B), the one previously considered to be moving at the speed v. In this case, the same reasoning as above (noting that $(-v)^2 = v^2$) leads to the opposite relation between time measures

$$\delta t = \delta t_0(1 - (v/c)^2)^{1/2}$$

We see then that just as we have an observer (A) whose life process tells him that he must have aged more than his travelling friend by a factor $(1 - (v/c)^2)^{-1/2}$ so, from the point of view of the latter man, the heart of the former will have beat a fewer number of times during one of his own time intervals, and the man who stayed on Earth will seem to have aged less (by the same factor). According to this analysis, if the clocks of both were initially synchronized when they were in the same inertial frame, then after they meet again in the same inertial frame each would assert that the clock of the other man was slow compared with his own clock. Or, if we could transfer all of the information to only one of these men (say the observer at the Earth space center), he would claim that he had aged both more and less than did the other man (the space traveller) by the time the trip had ended—a logical paradox!

An attempt to get out of the paradox is to note that, contrary to our assumptions, there is no symmetry in the motions of the two men. One of the men, the space traveller, changed inertial frames three times: leaving Earth, slowing down somewhere in outer space to turn around for the return journey and then slowing down to return to the original frame of reference (the "splash down"). If the logical paradox is to be removed and if, indeed, there is an asymmetric aging effect, it can only exist because of these periods of acceleration and deceleration. If the times taken to accelerate and decelerate are arranged to be very small compared with the times of uniform motion, proponents of the asymmetric aging effect assert that the periods of nonuniform motion, when the Lorentz transformation of equation 1 is not valid, contribute only negligibly to the *amount* of asymmetric aging, even though these short periods *cause* the effect. (An analogy might be a small bump on an emitting antenna causing a large distortion on a wavefront that is very far from the emitter.) The amount of aging asymmetry here still depends on the factor $(1-(v/c)^2)^{1/2}$, as it did in the earlier argument. (For a survey of this point and a complete bibliography, see reference 1).

The universe also counts

Many of us feel that this analysis does not settle the question, for it can still be argued that not only the velocity, but also the acceleration, is symmetrical with respect to the motions of the two men. If the space traveller is accelerating away from the observer on Earth, the Earth observer is equally and oppositely accelerating away from the space traveller. No explicit refutation of such symmetry between the accelerating motions has been demonstrated by the proponents of the effect. It appears then that, to the space traveller, the observer on Earth is the one who is changing inertial frames to make the round-trip journey that eventually brings him back to the inertial frame of the rocket ship. Each man will once again claim that the other has aged less after the journey is over.

This difficulty was recognized by Einstein and Reichenbach,[2–4] among others, in the early history of relativity theory. But they still claimed, as we have mentioned above, that there is a *bona fide* asymmetric aging effect because in reality there is more to the system than the two men who are comparing notes; there is also the entire universe. Now the result is that each man would agree that the traveller who left Earth must be the one who aged less during the round trip.

A mathematical analysis of the problem according to Einstein's prescription (see, for example, reference 8), supposes that two identical clocks, one belonging to the man A at the space center on Earth and the other belonging to the man B in the rocket ship, are synchronized before takeoff. After the countdown, the rocket engines exert a force F' for a time t_A', causing B to accelerate away from the launching site, until the rocket reaches a speed v. B's rocket ship then proceeds for a time $t_A/2$ to move away from A (and Earth) with the constant speed v until a second force F'' is applied by the rocket engines for a time t_A'', causing the rocket to slow down, turn around and proceed back toward A with the velocity $-v$. The ship then travels at this speed for $t_A/2$ seconds, until it comes near Earth. A third force, F''' then acts for t_A''' seconds to decelerate B's rocket until it splashes down on Earth, where A is waiting to meet B.

If the trip is arranged so that the times t_A', t_A'' and t_A''', during which F', F'' and F''' act, are very short compared with the time t_A of uniform motion, then according to special relativity theory the time of round-trip travel t_A, as measured by A, relates to the time of round-trip travel, t_B (as measured by A on B's clock) as

$$t_A = t_B(1 - (v/c)^2)^{-1/2} = t_B(1 + 1/2(v/c)^2 + \ldots) \qquad (2)$$

"Using Einstein's prescription, both A and B should agree that space traveller B is the one who ages less during the round trip."

According to this result, and the assumption that the physical aging process of A's body is linearly proportional to t_A and that of B's body to t_B, B would be younger than A after the journey by the factor $(1-(v/c)^2)^{1/2}$.

As we have noted, the apparent paradox appears when we let B record the various times, considering himself to be at rest and A, as well as the launching pad, to be moving. The same analysis as above gives the result.

$$t_A = t_B(1 - 1/2(v/c)^2 + \dots) \qquad (3)$$

If, with Einstein, we note that the two cases are not physically equivalent if the universe is taken into account, equation 3 must be modified. The modification comes during the period t_A'', when there is a reversal of motion for the return journey. The acceleration and deceleration periods t_A' and t_A''' at the beginning and ends of the round trip play no role (in this view), because A and B are at almost equivalent positions in the universe at these times; they are both close to Earth.

To handle the turn-around period, Einstein appeals to the principle of equivalence. Thus, to turn A (and the universe) around with respect to B, one can *equivalently* talk about a body that falls freely in a uniform gravitational field. If t_A'' is the time recorded by A on his own clock during this period, and if t_B'' is A's measurement on B's clock during the same period, then we can relate these two measurements according to the Doppler effect, which in turn depends on the relative velocity during this period.

According to the principle of equivalence, the appropriate equation of motion to determine this velocity is g = constant, where g is the acceleration due to gravity. The first integral of this equation gives the result $v'' = gt''$, where t'' is the time for the information signal to move between the two clocks at the turn-around point when they are, say, x cm apart. Then $t'' = x/c$, so that $v'' = gx/c$ is the relative velocity during this period. If we use this relation with the Doppler formula of special-relativity theory, we have

$$\begin{aligned} t_A'' &= t_B''[(1+v''/c)/(1-v''/c)]^{1/2} \approx t_B''(1+v''/c) \\ &= t_B''(1+gx/c^2) \end{aligned} \qquad (4)$$

Here the approximate equality corresponds to the assumption $v''/c \ll 1$.

The total distance travelled during the round trip is $2x$. Because t_B is much greater than t_B', t_B'' and t_B''', we can take $2x = vt_B$, from the point of view of B's clock, as observed by A. Because $2v$ is the total change in velocity at the turn-around point, in the time t_B'', we can set $g = 2v/t_B''$, according to the principle of equivalence. Combining equations 3 and 4, with v/c much less than one and $t_A' = t_B'$; $t_A''' = t_B'''$, we find

$$t_A = t_B(1 + 1/2(v/c)^2 + \dots) + t_B' + t_B'' + t_B'''$$

If we now assume that the periods of nonuniform motion, t_B', t_B'' and t_B''' are much less than t_B, then t_B is essentially the round-trip travel time, according to the measurement on B's clock (from A's view). With this approximation an overcompensation does indeed occur at the turn-around point (from the view of the space traveller's inertial frame) and, instead of equation 3, the times for the round trip are related according to the formula

$$t_A = t_B(1 + 1/2(v/c)^2 + \dots) \qquad (5)$$

Equation 5 agrees with equation 2, which gives the relation according to A's inertial frame. Using Einstein's prescription, then, both A and B should agree that the space traveller B is the one who would age less during the round trip.

Recall the assumptions used to derive this result: It is accurate to use the principle of equivalence together with a version of the transformations of special-relativity theory for the periods of relative nonuniform motion; v/c is much less than one; the mutual separation x between A and B at the turn-around point is sufficiently small that gx/c^2 is much less than one.

To order $(v/c)^2$,

$$1 + 1/2(v/c)^2 = (1-(v/c)^2)^{-1/2}$$

so that, to this order in v/c, equations 2 and 5 would be the same if we had inserted the factor from special-relativity theory $(1-(v/c)^2)^{-1/2}$, rather than the factor shown. Nevertheless when we take closer account of the times of nonuniform motion and carry out the above calculations to higher order than $(v/c)^2$, the series on the right sides of equations 2 and 5 do not necessarily add up to $(1-(v/c)^2)^{-1/2}$, nor do they necessarily agree with each other in these higher-order terms. Thus, although the use of the equivalence principle in this problem implies that A and B would agree that it is B who ages less, it is not clear that they would agree on the exact amount of asymmetric aging that had taken place. But to satisfy the principle of relativity, as we must, A and B must agree, both qualitatively and quantitively.

Quite apparent to me, then, is the need to reexamine, *from first principles,* whether or not relativity theory predicts the asymmetric aging effect. What follows here is my argument for rejecting the effect as being logically inconsistent with relativity theory itself, and an unambiguous mathematical proof of my contention, *without any approximation.*

Illogic of asymmetric aging

In Einstein's relativity theory, the distinction between the "proper" frame of reference and the "relative" frames is important. The "proper" reference frame of space–time coordinates is that frame in which we can directly relate to the real set of properties of the physical system, the properties of matter as explained in terms of its internal constitution. "Relative," on the other hand, refers to the space–time coordinates that relate to the physical properties of a system in motion relative to an observer's coordinate frame, as described by the observer. Thus, it *appears* to a stationary observer that a moving rod is shorter, in the direction of its motion, than it *really* is in the proper frame of reference.

If the stationary observer does not know about the theory of relativity, then he would disagree with an observer who moves with the rod about the rod's real length. But if he knows about relativity theory and the correct way to translate from one reference frame to the relatively moving frame, both observers would agree on the *real* length of the rod, the length that is independent of any travelling observers who happen to be looking at the rod. This is the "proper" length of the rod: the length determined only by the locations at which electrostatic forces within a system of about 10^{24} molecules establish their equilibrium positions in their proper space.

Similarly, to specify a time measure in relativity theory, we must specify the motion of the observer relative to the proper frame of reference. That is to say, an observer who is sitting on a clock or measuring his heartbeat rate, for example, can relate directly to the proper time measure. But an observer who is in motion relative to the clock does not measure the proper time interval; that is, he does not measure the time interval that corresponds to a certain amount of aging and is a consequence of the internal forces that constitute the time-measuring device. The travelling observer would, however, be aware of this discrepancy if he is aware of relativity theory and of his motion relative to the clock.

The idea here is that space and time, according to the theory of relativity, are no more than the elements of a language that expresses the laws of nature in one frame of reference or another. The fundamental assertion of the theory is that the law of nature itself must be the same to all possible observers, irrespective of their relative motions. This is the *principle of relativity*.

But there is something special about the particular set of space-time coordinates that is called "proper": These coordinates describe aspects of a physical system that are independent of the motion of any observer. Thus, to know the actual aging of some physical entity, one must convert the measured time interval of his own clock (if he is in motion relative to the observed system) to the time interval that would be measured if he were in the proper frame. For the "proper time," not the "relative time," describes the internal physical processes within the observed system that are responsible for the physical effect we call "aging."

This principle is independent of whether the physical aging refers to a living organism, a piece of radioactive metal or a common clock. For living organisms, the aging effect is the set of electromagnetic forces that manifest themselves as complicated chemical–physical processes of irreversible cell decay. For radioactive metal, the nuclear, electromagnetic and weak interactions, exerted by the constituent nucleons and mesons on each other, are responsible for the observed deexcitation of these nuclei at certain fixed rates. And in the common clock the mechanical properties of its spring (stiffness, length) account for the "aging" measured on its face.

Relativity theory does not claim that any of these forces or their effects on aging are in any way affected by the relative motion of some observer who may be looking at the considered system from a moving platform. It claims only that the laws of nature, which explicitly describe these forces, must be the same in all relatively moving space-time frames. It is still true of course that if one of these different types of clock should be in motion relative to some observer, he would report that its time measure does not agree with his own. But when the clock enters his own inertial frame, coming to rest with respect to him, the observer would then be in the proper frame of the clock and could measure directly the "true" time measure—the time measure determined by the internal physical forces of the clock that cause its aging.

It appears to me then that two identical clocks, synchronized in a single proper frame of reference at some initial time, should be synchronized at all future times of observation, even though they may not appear to be synchronized unless they are again in the same proper frame of reference. This conclusion is consistent with the philosophic view of space and time in relativity theory: Coordinates are no more than language elements used by the observers of natural phenomena to express physical laws, and the time coordinate is not itself a physical process, such as aging.

This conclusion assumes that there were no physically damaging effects on the internal mechanism of the clock that travelled (see references 2 and 9 for the opposing view); in any case, the majority who believe that the clocks in this problem will lose their synchronization claim that the loss is due strictly to the history of their relative motion, independent of forces that might act on their internal mechanisms. According to the view of time as a subjective parameter in the description of a physical law, I see this majority claim as logically incompatible with the axioms of relativity theory. To accept this majority conclusion is to me a regression to the classical view in which time is indeed an absolute measure, with its own physical manifestations.

Mathematical argument

I now pose the clock question in precise mathematical language. The differential increment of proper time, as measured by some entity that moves with the clock (at the origin of the moving frame) is ds. The answer to the question: "How much time has passed during some finite increment of aging from the *proper time* s_1 to the *proper time* s_2?" is given by the value of the line integral

$$\oint_{s_1}^{s_2} ds$$

or by any quantity in one-to-one correspondence with this line integral. The particular path of this clock in space-time is denoted by C. For example, if we refer to the aging process within our own inertial frame, then C is a straight-line path. On the other hand, the aging may take place along a different sort of path, C', in which a clock is accelerated and decelerated relative to C (for at least part of the path) such that the initial and final proper times that correspond to the measured aging are the same proper times, s_1 and s_2, of the first clock. In this case the total time for this aging process would be the value of the line integral

$$\oint_{s_1}^{s_2} ds$$

The question about clock synchronization then reduces to the question about whether or not the values of these two line integrals are the same. That is, for arbitrary paths of two mechanically identical clocks in space-time, synchronized in the same inertial frame at the initial proper time s_1, is the value of the line integrals path independent? Does

$$\oint_{s_1}^{s_2} ds = \oint_{s_1}^{s_2} ds? \qquad (6)$$

To answer this question, we need not know the explicit features of the space-time metric. We shall see that we only need know whether or not the metric field is single valued and analytic at all space-time points where the physical aging process is occurring.

Because in relativity theory, an *arbitrary path* entails a noninertial co-

" . . . space and time, according to relativity theory, are no more than the language that expresses the laws of nature in one frame of reference or another."

ordinate frame, we must start this analysis with the expression from general relativity theory for the invariant metric ds. As is the convention, we start with the space–time points defined within a Riemannian manifold, with the corresponding squared differential metric

$$ds^2 = g^{\alpha\beta}dx_\alpha dx_\beta \qquad (7)$$

where x is a four-valued point x_0, x_1, x_2, x_3 and $g^{\alpha\beta}(x)$ is the metric tensor field that solves Einstein's field equations. (The usual notation is implied, where the summation signs over time and space indices, $\alpha,\beta = 0,1,2,3$, are deleted.)

To prove that the total aging of the clock is path independent, we must use ds rather than ds^2. Although ds is usually determined by simply taking the square root of ds^2 from equation 7, this method will not do for the problem at hand because of an ambiguity in the sign of the square root of ds^2 at the different space–time points x and because of the information that is thereby lost. We use instead a more general (and unambiguous) form for ds, the quarternion factorization of ds^2, yielding the form[10]

$$ds = q^\alpha dx_\alpha \qquad (8)$$

where the metric field q^α (x) is a four-vector field in which each vector component is, algebraically, a quaternion. Because a quaternion has four elements, q^α is a 16-component field. Here ds is expressed as a sum of four quaternions and is therefore represented by a quaternion rather than a real number. The correspondence with the *real number* field ds^2 in equation 7 is then obtained from equation 8 by multiplying 8 by its quaternion conjugate

$$ds\bar{ds} = 1/2(q^\alpha\bar{q}^\beta + q^\beta\bar{q}^\alpha)dx_\alpha dx_\beta (=) g^{\alpha\beta}dx_\alpha dx_\beta$$

The field equations in q^α were derived by me (see reference 10). They correspond to 16 relations at each space–time point. From an iteration of these equations in q^α, they can be reexpressed in a second-rank tensor form. The latter 16 relations may then be grouped into the sum of a symmetric tensor part (10 relations) and an antisymmetric tensor part (6 relations). The symmetric tensor part has been shown[11] to be in one-to-one correspondence with the full structure of Einstein's field equations.

All of the physical predictions, then, that are implicit in Einstein's equations are also implicit in the quaternion field equations, although the latter contain predictions that have no counterpart in the tensor formalism. Thus any of the physical consequences of equation 7 must also be contained in the consequences of equation 8 for the invariant metric. The extra predictions of the quaternion formalism entail features of space–time that are not involved in the problem at hand (for example, polarization and bound-state effects and the effects of interacting "currents"). The results to be determined from the quaternion form of ds are also implicit in the tensor formalism, but because of the lack of ambiguity in the quaternion form, compared with the square root of ds^2 from equation 7, only the quaternion form can be used with the method below to answer our question.

In the special frame of reference, in which the clock's motion can be described in a two-dimensional plane, that is, in a slice of space–time where any two x, say x_0 and x_3, are held fixed, $dx_0 = dx_3 = 0$. In this special reference frame, we call $q^1 = Q^1$, $q^2 = Q^2$ and $ds = dS$. Any closed-path line integral of ds then has the form

$$\oint dS = \oint(Q^1dx_1 + Q^2dx_2) \qquad (9)$$

Because ds is generally an invariant of the Riemannian space–time, a conclusion of the analysis that this closed-path line integral is zero is transformable to any 4-dimensional space–time. The transformability implies that the closed-path line integral of ds in the general reference frame would also necessarily be zero.

Recall that the quaternion field q^α (and therefore Q^1 and Q^2) is a set of 2-dimensional Hermitian matrices. Each of the quaternions in equation 9 is then a function of four real fields; let us call them f_μ^α. The closed-path integral then stands for a 2-dimensional matrix of closed-path integrals, in which each component (real) field has the form

$$\oint(f_\mu^1(x_1,x_2)dx_1 + f_\mu^2(x_1,x_2)dx_2) = \mathrm{Re}\oint f_\mu(z)dz \qquad (10)$$

The right-hand side of equation 10 has

been chosen, for convenience, to describe the variables on the left in terms of a complex plane, where

$$f_\mu^1(x_1,x_2) - if_\mu^2(x_1,x_2) = f_\mu(z)$$

$z = x_1 + ix_2$, and Re stands for the real part of the closed-path integral. Of course, the functions in equation 10 also depend on x_0 and x_3; these are suppressed here, because they are not involved in the integration.

According to the Cauchy theorem of complex variables, if $f_\mu(z)$ is a single-valued function of z that is analytic at all points in and on the closed curve of integration, then

$$\oint f_\mu(z)dz = 0$$

The real and imaginary parts of this integral must then be separately zero, so that the right-hand side of equation 10 is zero if each of the four fields that make up the quaternion variable are separately single valued and analytic. Under these conditions, then, the closed-path integral in equation 9 stands for a null matrix. Because $dS = ds$, it follows that in any space-time frame of reference within the Riemannian manifold, for a metric field that is single valued and analytic in that domain where the aging is described (away from possible singularities), we find the desired result

$$\oint ds = \oint q^\alpha dx_\alpha = 0 \qquad (11)$$

Note that the vanishing of the *closed-path* integral in equation 11 does not imply that the total aging of some physical entity is zero. The closed path, which assigns polarity to line segments (for every positive segment of a *closed path* there is a corresponding negative segment) is only a mathematical device to investigate a particular feature of the space-time. The vanishing of the closed-path integral is simply a way of expressing the equivalent mathematical feature of the space-time that any line integral of ds, evaluated between two arbitrary space-time points, is path independent.

$$0 = \oint q^\alpha dx_\alpha = \oint_{s_1(x)}^{s_2(x)} q^\alpha dx_\alpha + \oint_{s_2(x)}^{s_1(x)} q^\alpha dx_\alpha$$

$$\oint_{s_1(x)}^{s_2(x)} q^\alpha dx_\alpha = -\oint_{s_2(x)}^{s_1(x)} q^\alpha dx_\alpha = \oint_{s_1(x)}^{s_2(x)} q^\alpha dx_\alpha \qquad (12)$$

where $s(x)$ refers to the space-time values at the end-point proper times s_1 and s_2 of either one of the clocks, say the one that stays on Earth.

We have seen that the total time for aging of some physical body, between the proper times s_1 and s_2 is independent of the path traced out in space-time. If C is a path in an inertial frame of one clock (say on Earth), and if C' is any other path, not inertial relative to path C, traversed by a second (identical) clock, then the total time during which the second clock ages (the right-hand side of equation 12, measured from the initial to the final proper times (s_1 to s_2) of the first clock's aging process) is exactly equal to the total time of aging of the first clock (the left-hand side of equation 12).

Should future experimental evidence refute this conclusion, then according to the analysis we have just seen, Einstein's theory of relativity would be refuted.

References

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