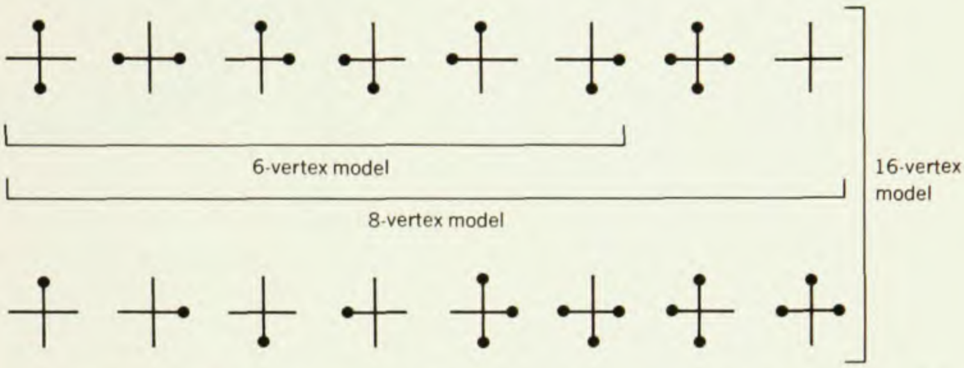
Solution of eight-vertex model excites critical-point theorists

A difficult problem in two-dimensional statistical mechanics has been solved by R. J. Baxter of the Australian National University in Canberra (*Phys. Rev. Lett.* **26**, 832, 1971). The so-called "eight-vertex" model that he solved contains as special cases the square lattice Ising, dimer, ice, F and KDP (potassium dihydrogen phosphate) models. Baxter's elegant and beautiful solution has challenged one of the widely held precepts of critical-point theory, namely that the character of the critical behavior is independent of the substance provided the symmetry properties of the Hamiltonian are the same.

In a companion paper,[1] Baxter develops a solution of the general case of the Heisenberg linear chain of spins, a chain with unequal couplings in the x, y and z directions.



Sixteen-vertex problem involves all the ways you can arrange zero through four dots around a point. The eight-vertex problem allows only even numbers of dots. The six-vertex problem does not allow zero and four dots. R. J. Baxter has solved the eight-vertex model, which contains as special cases the square lattice Ising, dimer, ice, F and KDP models.

The ice problem uses a six-vertex model, in which you think of each oxygen vertex of the ice lattice as having four nearest-neighbor oxygen atoms. There are six ways of arranging two hydrogen atoms around each oxygen atom. In the figure the dots represent the hydrogen atoms. In the ice problem all the hydrogen configurations or vertices have the same energy. On the other hand, in the problem for ferroelectrics or antiferroelectrics, the different configurations have different values for energy.

The eight-vertex problem allows zero and four dots. Besides representing hydrogen bonds, these dots can stand for any physical quantity, such as spin, that exists at the interstices of a lattice. An even more general problem is the 16-vertex problem, which involves all the ways you can arrange zero through four dots around a point. In the eight-vertex case you allow only an even number of dots around each vertex. The dots are arranged at the corners of a square, N-by-N lattice. One assigns energies $\epsilon_1, \ldots \epsilon_8$ to the vertex configurations and the problem is to evaluate the partition function

$$Z = \sum \exp\left(-\sum_{j=1}^{8} N_j \epsilon_j / k_B T\right)$$

where the summation is over all allowed configurations of dots on the lattice, N_j is the number of vertices of type j, k_B is the Boltzmann constant and T is absolute temperature.

History. Most crystals essentially do not have much entropy left. But when the entropy of ice was first measured, it was found that it had a lot of residual entropy left, that is, a lot of residual disorder. Linus Pauling proposed a model for ice where the hydrogen atoms essentially formed the bonds between the oxygens in the water molecules. Each oxygen would have two nearby hydrogens and two far-away hydrogens. You then have a combinatorial puzzle, the number of ways of doing the arranging, which measures the entropy. Then John F. Nagle (Carnegie-Mellon

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ISR experiment shows scaling behavior

One of the first experiments to be done at the new CERN Intersecting Storage Rings has verified certain scaling laws developed by Richard Feynman (Cal Tech) and by C. N. Yang (State University of New York at Stony Brook) and his collaborators. The experiment was just reported by Lazarus G. Ratner (Argonne), Richard J. Ellis and Gianni Vannini (University of Bologna), Bryce A. Babcock, Alan D. Krisch and Jabus B. Roberts (University of Michigan) in the 5 July issue of *Physical Review Letters.*

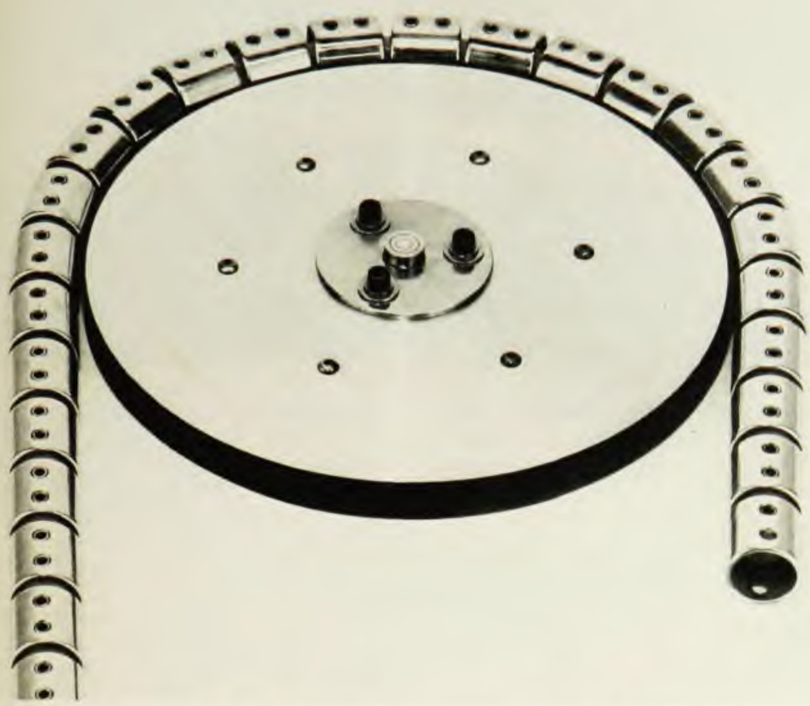
The group studied inelastic inclusive processes, that is reactions in which

$$p + p \rightarrow \pi^+ + \text{anything}$$
$$p + p \rightarrow p + \text{anything}$$
$$p + p \rightarrow K^+ + \text{anything}$$

They varied the incident proton momentum in each ring between 15.3 and 26.5 GeV/c; the equivalent laboratory momentum is 500–1500 GeV/c. Each ring was filled to an intensity of about 4×10^{13} protons, and then the beams were coasted for 6 to 12 hours with a loss rate that was often as low as 0.1% per hour. Vacuum was maintained at 5×10^{-11} torr.

A 45-meter-long spectrometer detected outgoing particles. Two threshold gas Cerenkov counters were used in coincidence and anticoincidence to tag the particles as pions, kaons or protons.

One might guess that the single-particle distribution observed in the ISR experiment would depend on the total energy of the reaction, P_l (the component of the momentum of the produced particle parallel to the initial direction) and P_t (the component of the produced par-



Pelletron charging chain is new way to transfer charge in an electrostatic accelerator. The metal cylinders of pellets pass over 30-cm-diameter pulley.

riers. In the Pelletron if at one instant 500 pellets are carrying charge in the uniform field of the column and this number suddenly changes to 501 as a pellet enters the terminal, one might expect a current change of one part in 500, which with typical values for terminal capacitance and charging current would give a terminal ripple of about 1 volt. Actual ripple due to discrete charges, Herb said, will be even lower because the column field at terminal and ground ends tapers over several pellet lengths from full value to zero.

Pellet-chain charging offers improved dependability, voltage stability, freedom from attention, and freedom from dust and lint, according to Herb. An unexpected thing happened with a 2-MV test generator, when the motor driving the chain was turned off with its terminal at +2MV and with voltage on the charging inductors. The chain came to a stop, turned backward at relatively high speed until the generator was discharged and then continued until a negative charge was deposited on the terminal. "We probably had the first 2-MV motor," he remarked.

Pelletron accelerators are built of 1-MV modules. Accelerating tubes are all metal and ceramic with a metal bonding agent. When the modules are stacked there is no indication of departure from voltage additivity. An eight-unit accelerator is now being assembled at the University of Sao Paulo. A 20-MV-terminal machine, which yields 40-MeV protons, has been designed, Herb said, and several labs are trying to obtain funds to buy such a device.

The University of Wisconsin has installed a Pelletron charging system in a High Voltage Engineering Type EN tandem accelerator. So far there have been difficulties that were caused by mechanical failures. When the system was operating properly, however, the voltage stability was much better than with the belt-charging system. —GBL

Eight-vertex model

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University) improved the numerical accuracy of Pauling's approximate estimate of the residual entropy.

Elliott Lieb (MIT) was able to solve exactly the two-dimensional version of the ice model. Then he and independently Bill Sutherland, C. N. Yang (State University of New York at Stony Brook) and C. P. Yang (Ohio State University) extended this class of ice models to various ferroelectric models, such as the KDP model and others, all of which obey the same rules for putting in hydrogens as does ice. You can associate different energies with the hydrogens, however.

Baxter solution. In solving the eight-vertex problem Baxter uses the approach introduced by H. A. Kramers, which was originally used by Lars Onsager to solve the Ising model. This matrix method allows you to reduce two-dimensional models to one-dimensional quantum-mechanical spin models. You then use the ground-state wave function and the ground-state energies for the one-dimensional situation to find the free energy and correlation functions of the original two-dimensional problem.

Baxter effectively guessed the form of the eigenfunctions in the transfer-matrix formulation of the problem, and he showed that his guessed eigenfunctions were correct. The original clue to solving the problem is to use the so-called "Bethe Ansatz" approach, which was developed for the Heisenberg linear chain of spins. You say that you think the ground-state wave function has a certain form. You try it, and it has some undetermined phase shifts. By substituting the function in the equations, you can get equations for phase shifts, and then obtain an integral equation with something like a density of phase shifts, from which you can then obtain the exact ground-state energy.

Baxter's solution abounds in classical elliptic integrals and elliptic functions; one theorist calls it a masterpiece of 19th-century mathematical analysis.

Critical points. Baxter's solution says that when an eight-vertex model has a phase transition, then in general the free energy has a branch-point singularity. The exponent of the singularity can range continuously from one to infinity and depends on the energies involved in the eight vertices. This is the result that has surprised many workers in critical phenomena. Ever since Onsager solved the two-dimensional Ising model, it appeared that the character of the singularity was independent of the details of the lattice structure. Onsager found that the specific heat varied as $\log|T-T_c|$. The model could be solved for a number of different lattice arrangements.

The general idea that singularities don't depend on fine details has been borne out by extensive experimental studies on real three-dimensional systems and by numerical and series extrapolation expansions on theoretical three-dimensional systems. The shape of the specific-heat curve might be a function of $(T - T_c)$ to some power (called the "critical exponent") instead of a logarithmic function, but it was believed to be independent of whether you are looking at carbon dioxide or xenon, for example. If you were talking about the three-dimensional Ising model you didn't have to say that it's a face-centered cubic lattice or a simple cubic lattice, or whether the interactions are the same in different directions, or different. This assumption is known as "universality" or "smoothness."

At the informal Yeshiva University meeting on statistical mechanics, F. Y. Wu[2] (Northeastern University), and independently Leo Kadanoff and Franz Wegner[3] (Brown University) reported that they have shown that the eight-vertex model is equivalent to the two-dimensional Ising model with next-nearest-neighbor interactions and special four-spin interactions.

Speculations. As Michael Fisher (Cornell University) suggested to us, if you break the symmetry of the eight-vertex model by taking some of the interactions to be slightly different, you go back to the ordinary Ising model with a logarithmic specific heat. You get two logarithmic specific heats close to each other, a kind of double transition; where they come together something special has to happen, and this special point is what Baxter has found. If this is the case, Baxter's special critical point resembles conceptually the tricritical point observed in He^3-He^4 mixtures, Fisher said. You can think of it either as the end point of a line of critical points or a place where lines of two critical points come together.

Kadanoff reported that he and Wegner had shown[4] that although the Baxter work calls universality into question, the widely believed concept of scaling holds in the neighborhood of a point at which the Baxter solution reduces to the Onsager solution. Kadanoff feels that the direct experimental implication of Baxter's solution is slight. However, now that we know the solution has critical exponents that depend continuously on a parameter, he said that we must look at real three-dimensional phase transitions to see whether they in fact do have this property.

Recently A. M. Polyakov and A. A. Migdal (Landau Institute in Moscow) have emphasized the analogy between critical phenomena and relativistic quantum field theory.[5] Last year Kenneth Wilson (Cornell) pointed out that there is a field theoretic model, the Thirring model, which has continuously variable exponents[6] (like the Baxter model). In field theory these exponents are called "anomalous dimensions." Because of the analogy to critical phenomena, the existence of the Thirring model was a warning that universality might break down, but before Baxter's work apparently no one took this warning very seriously.

The flexibility inherent in the variable exponent offers great opportunity for further theoretical efforts. —GBL

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Eskimo roller. Actually James C. Sindelar is not an eskimo but an associate professor of electrical engineering at New Hampshire Technical Institute. He wrote the winning entry in the competition, sponsored by the Old Town Canoe Company, advertised in the January issue of *physics today;* our photograph shows him in the slalom kayak that was awarded as the competition prize.

The problem posed for the competition was: How does a kayaker who finds himself head down underwater succeed in righting himself? Similarly, how can one learn to execute the complete "Eskimo Roll"—a deliberate 360-degree rotation from head-up to head-up?

Sindelar's entry, an 1800-word treatise on the Eskimo Roll, included historical background, an analysis of the dynamics of the motion, and a discussion of the effect of boat hull shape on the ease of carrying out the maneuver.

Apparently Alaskan and Greenland eskimos originally learned the trick as a means of survival in Arctic waters; later they developed variations of the roll as a sport. Nowadays the sport has been taken up by US and European whitewater kayakers.

For the physics of the motion, Sindelar divides the 180-deg righting movement into two parts—a 90-deg motion that brings the kayaker's body up to the water level, followed by another 90-deg motion up to the vertical. He shows that maximum efficiency is obtained if the body is extended at first, to derive greater torque from the sweeping paddle, and then moved closer to the boat's roll axis to minimize the moment of inertia (just as in the classical example of the spinning ice skater who draws in arms and legs to speed up the spin towards the end).

The shape of the boat, Sindelar says, has a considerable effect on the effort required to execute the roll. Perhaps surprisingly, he finds that the less stable boats often prove more difficult to roll than do the stable designs. This is because greater stability implies a center of gravity that rises higher above its normal minimum height when the boat is tipped 90 deg from its equilibrium position. So a stable boat stores potential energy in this position, just when the kayaker needs an extra push to get him over and when his available torque with the paddle happens to be at a minimum. —JTS

An applications-oriented approach to atmospheric science

The Atmospheric Sciences and Man's Needs: Priorities for the Future, a recent report of the Committee on Atmospheric Sciences of the National Academy of Sciences, discusses objectives and priorities for US atmospheric research in the 1970's.

The report calls for an applications-oriented approach to atmospheric science, with emphasis on improved weather prediction and experimental weather modifications. One recommendation of the report is the continued support of GARP (the Global Atmospheric Research Program), "to extend the range of useful prediction into the one- to two-week period." As well as increasing the accuracy of long-range weather predictions, the calculating and measuring equipment used in the global program will have application to global modeling techniques that should lead to an improved understanding of air-pollution dynamics.

If the recommendations of the report are adopted, local weather forecasting may be improved by the "Pilot Local Weather Watch," a system that would automatically issue radio and television warnings to people in areas endangered by sudden storms.

The report was prepared by a group of 55 atmospheric researchers during the summer of 1970 and the ranking of priorities was established by a steering committee under the direction of Robert G. Fleagle of the University of Washington. Copies can be obtained from the Printing and Publishing Office, National Academy of Sciences, 2101 Constitution Avenue, N.W., Washington, D. C. 20418, for $3.25 per copy. □