# CAN EQUATIONS OF MOTION BE USED IN HIGH-ENERGY PHYSICS?

## The formalism that has been so successful for classical physics might lead to a useful, as well as aesthetically pleasing, theory of particles.

P. A. M. Dirac

THE PHENOMENA of high-energy physics have stimulated the development of several new mathematical approaches to calculate and explain the experimental results. Many of these approaches bear little relation to methods used in other areas of physics and many have incomplete or unsatisfactory aspects to them. They have been used with varying success. Methods based on the equations of motion, so necessary for low-energy physics, have been largely abandoned as being intractable to this latest branch of physics. Yet if we believe in the unity of physics, we should believe that the same basic ideas universally apply to all fields of physics. Should we not then use the equations of motion in high-energy as well as low-energy physics? I say we should. A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data.

Let us understand precisely what is meant by equations of motion. One must introduce a set of quantities $A$, of any mathematical nature, to describe the physical state at a certain time. The equations of motion are then

$$dA/dt = \text{function}\ (A) \qquad (1)$$

By integrating the equations, one can calculate $A$ at a later time in terms of the initial set $A$.

The determinism implied by the equations of motion does not hold generally in the atomic world. But there exists a quantum mechanics, based on equations of motion, valid for low energies, with determinism holding between observations and getting disturbed only by observations. The question is whether there exists a similar theory based on equations of motion that is also valid for high-energy physics?

Up to the present, equations of motion have not had any significant success in high-energy physics, apart from the limited domain of electrodynamics. The difficulties of applying equations of motion in a theory that must obey relativity are so serious that many theoretical physicists are inclined to give up the attempt and build theories independent of equations of motion.

### *S-matrix theory*

Some physicists point out—quite justifiably—that the quantities involved in equations of motion, namely dynamical variables referring to one instant of time, are not connected very closely with experimental results, and they say that a theory should be expressed in terms of quantities directly connected with observation. This is a pretty good argument, and is one that led Werner Heisenberg in 1925 to construct matrix mechanics, which evolved into our present quantum mechanics.

In the case of high-energy physics one is concerned with calculating the probabilities for emission, absorption and scattering of particles. These probabilities are given, if one assumes the same general principles that work so well for low-energy physics, by the squares of the moduli of certain numbers, called probability amplitudes. The probability amplitudes, collected

After 37 years as professor of mathematics, P. A. M. Dirac, who won the 1932 Nobel prize for his relativistic theory of the electron, has retired from the University of Cambridge. However, Dirac has not retired from physics. He is spending this academic year at the State University of New York at Stony Brook and at the Center for Theoretical Studies at Coral Gables, where he is "mainly concerned with putting quantum electrodynamics on a logical basis."

## "A theory with mathematical beauty is more likely to be correct than an ugly one that fits some mathematical data."

together, form the S-matrix. Thus a knowledge of the S-matrix would provide all the information needed in high-energy physics.

If one had equations of motion, one could integrate them to get the S-matrix. But perhaps the S-matrix exists independent of the existence of the equations of motion. This belief, which has had a great deal of success, defines a school of physics, the "no-equations-of-motion" school. One knows some properties of the S-matrix from general physical principles and one can get a great deal more information about it by incorporating experimental results. The school hopes that ultimately sufficient information will be obtained to determine the S-matrix completely.

In spite of the progress of the S-matrix school, I believe that high-energy physics should be based on equations of motion because they are so necessary for low-energy phenomena. High-energy physics forms only a small fraction of the whole of physics. The theories of most fields, such as solid-state physics, spectroscopy of atoms and molecules, and chemical physics, are based, fairly satisfactorily, on equations of motion. We believe in the unity of physics. The equations of motion that are so successful for most of physics cannot be simply discarded for one branch of physics. Although these equations may need modification, perhaps involving different kinds of variables, one would still expect to retain the basic structure of equation 1. The result would be differential equations in the time, which one has to integrate to get results comparable with experiments.

### *Lorentz invariance*

A theory based on the equations of motion, in which the time is treated differently from the space coordinates, is not manifestly Lorentz invariant. This does not imply that the theory is wrong. We require the results of the theory to be Lorentz invariant, and we would have to prove that they are Lorentz invariant before the theory could be accepted as correct. The proof might be quite an involved one, but that would not matter.

Further it may be that the requirement of Lorentz invariance applies only to a complete theory that takes into account all the particles of physics and all the interactions among them. An incomplete theory, restricted only to certain particles, need not then be a Lorentz invariant theory. Present-day theories are all incomplete. Quite likely we do not yet know all the elementary particles that exist. There may not be a need to insist on Lorentz invariance for present-day theories.

If one believes that equations of motion should apply to high-energy physics, the natural way to proceed would be to take the equations that are successful for low-energy phenomena and try to develop and generalize them to make them apply to higher and higher energies. There is no need to require accurate Lorentz invariance during the procedure. We should require approximate Lorentz invariance in the application to the lower energies, and should strive gradually to increase the accuracy.

There is here an essential difference in the mode of approach from the S-matrix theory. In the latter, one has Lorentz invariance holding initially and at all stages of the development.

### *Quantum electrodynamics*

Low-energy physics is governed by quantum electrodynamics, which deals with charged particles interacting with the electromagnetic field. One would expect it to apply to all physical processes with energies up to a few hundred MeV, where the possibility of creation of other particles arises. One must therefore begin by establishing the equations of quantum electrodynamics in a satisfactory form. A number of problems arise here.

Maxwell's theory, together with the relativistic theory of the electron, provides definite equations of motion. One may proceed to solve them by a perturbation method, treating the interaction between the electrons and the field as small. The solutions are expressed as power series in the coupling constant $e^2/\hbar c$, a small number. However, one soon runs into divergent integrals.

Willis Lamb, Hans Bethe and others proposed rules for discarding the infinities from the equations so as to leave finite residues. Their results explained certain physical effects, the Lamb shift and the anomalous magnetic moment of the electron, with great accuracy. The usual quantum electrodynamics resulting from this procedure has satisfied most physicists. I do not find it at all satisfactory.

If one is to build up one's theory from equations of motion, one should use them according to the standard laws of mathematics and neglect only quantities that are small, not infinities. What is the situation when one does keep to standard mathematics?

Because of the infinities in quantum electrodynamics the equations of motion have no solutions, and they must be modified. The infinities come from the high-energy processes for which the theory cannot be valid because it does not take into account the other particles that then come into play. As we do not know enough about the other particles to be able to bring them into equations of motion, we are forced to cut out the high-energy processes altogether in order to proceed. The infinities are then removed but Lorentz invariance is destroyed. This latter failure is a lesser evil than abandonment of standard mathematics.

### *Formulation of the method*

The dynamical variables involved in this work consist of the operators of creation and destruction of electrons and photons in various states. Let $\eta_n$ denote the creation operators, where $n$ refers to a stationary state of some isolated particles. The destruction operators are then $\eta_n{}^*$. The Hamiltonian $H$ is of the form

$$H = E + V$$

where $E$ is the proper energy of all the particles. It is expressed as

$$E = \Sigma\omega_n\eta_n\eta_n{}^*$$

where $\omega_n$ is the energy of a particle in the state $n$. $V$ is the interaction energy and may be written as some power series in the $\eta$'s and $\eta^*$'s. The perturbation procedure involves treating $V$ as small.

There is a state $|0\rangle$ with no particles present, satisfying

$$\eta_n{}^*|0\rangle = 0$$

for all $n$.

Any state $|P\rangle$ can be expressed as

$$|P\rangle = \psi(\eta)|0\rangle \qquad (2)$$

where $\psi(\eta)$ is a power series in the creation operators $\eta$. If $|P\rangle$ is normalized, the squares of the moduli of the coefficients in $\psi(n)$ give the probabilities that particular numbers of particles are present in particular states.

In the Schrödinger picture $|P\rangle$ varies according to

$$i\hbar d|P\rangle/dt = H|P\rangle$$

This equation of motion calculates how $|P\rangle$ varies with the time. If we write

$$|P\rangle = \psi(\eta,t)|0\rangle$$

we can calculate how the probabilities vary with $t$.

Suppose at $t = 0$, $|P\rangle$ is a no-particle state.

$$|P\rangle_{t=0} = |0\rangle$$

We have then

$$i\hbar(d|P\rangle/dt)_{t=0} = H|0\rangle = V|0\rangle$$

This expression does not vanish. Thus, because $V$ contains creation and destruction operators, we find that particles are being created. After a time $|P\rangle$ will no longer be the no-particle state with which we started. The no-particle state is therefore not stationary.

### *The vacuum state*

It is customary to define the vacuum state $|v\rangle$ as the stationary state of lowest energy. It satisfies an equation of the form

$$i\hbar d|v\rangle/dt = \lambda|v\rangle$$

where $\lambda$, a real number, assumes its minimal values. The definition is reasonable provided all departures from the vacuum state involve an increase in energy. The vacuum state does exhibit this property for all the known fields of physics except the gravitational field, which is of no importance in atomic physics.

We are thus led to think of the vacuum state as something very different from the no-particle state. According to equation 2, one can be expressed in terms of the other.

$$|v\rangle = \psi_0(\eta)|0\rangle$$

where $\psi_0$ varies with $t$ according to the law $e^{-i\lambda t/\hbar}$.

If we are given $H$ with a suitable cutoff to avoid the infinities, we might try to calculate $\psi_0$ and get the probabilities for particular numbers of particles existing in the vacuum state.

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The calculation would be very difficult because a perturbation method cannot be used, the later terms in the perturbation expansion being more important than the earlier. Although the calculation should be possible in principle, it would not be very useful because the result would depend strongly on the cutoff. As long as we do not know just where or how the cutoff should be introduced, only results that are insensitive to the cutoff can be significant.

Perhaps one should try to build up the theory without knowing $\psi_0$, especially because it is only the departure from the vacuum rather than the vacuum itself that interests the experimenters.

A state that departs from the vacuum may be represented by $K|v\rangle$, the result of some operator $K$ applied to the vacuum state. Schrödinger's equation gives

$$i\hbar \frac{d}{dt}(K|v\rangle) = HK|v\rangle \qquad (3)$$

or

$$i\hbar \frac{dK}{dt}|v\rangle = HK|v\rangle - i\hbar K\frac{d}{dt}|v\rangle = (HK - KH)|v\rangle$$

This equation is satisfied provided we choose $K$ so that

$$i\hbar \frac{dK}{dt} = HK - KH \qquad (4)$$

which is a Heisenberg equation of motion. If we can get solutions of equation 4, they will provide us with solutions of the Schrödinger equation 3 even though we do not know $|v\rangle$.

### *Interpretation of the solution*

When we have a solution of the Heisenberg equation 4, how do we use it? We need some physical interpretation. According to the standard rules for the physical interpretation of quantum mechanics, we ought to express $K|v\rangle$ in the form of equation 2

$$K|v\rangle = \psi_1(\eta)|0\rangle$$

and calculate $\psi_1$ at any time. The probabilities that various particles are present could then be calculated from the wave function $\psi_1$. Using the wave function $\psi_0$ to compare these probabilities with those for the vacuum state, we should see what extra particles or lack of particles exist for the state $K|v\rangle$. However, this method of interpretation requires a knowledge of $\psi_0$ and cannot be used under present conditions.

I would like to propose another method of interpretation. Let us form $K|0\rangle$, express it as

$$K|0\rangle = \psi_2(\eta)|0\rangle$$

and calculate $\psi_2$ at any time. After it is normalized, $\psi_2$ determines the probability distribution of particles. No knowledge of $\psi_0$ then is needed. One would need a different normalizing factor for each determination of the probability distribution.

Although this method of physical interpretation is not in agreement with the standard laws of quantum mechanics, it still appears reasonable as a stop-gap procedure for use at the present time. Effectively, it smooths out the complicated effects of vacuum fluctuations.

One may use this method for calculating the Lamb shift and anomalous magnetic moment. The operator $K$ would correspond to the creation of an electron in a static electric or magnetic field at a certain time. The solution of equation 4 is then insensitive to the cutoff. After carrying through the whole calculation in a logical manner, one departs from accepted principles only in the use of a new method of physical interpretation.

Taking $K$ to be the operator of creation of a photon at a certain time, one could try to make a similar calculation. In this case the solution of equation 4 depends strongly on the cutoff. It would correspond to the photon having a large rest mass, an infinite one if there is no cutoff. This serious fault in the theory must be corrected by a change in the Hamiltonian. Before we can extend this theory to higher energies we must be able to include all other particles and interactions. □