UNITS FOR LOGARITHMIC SCALES

Even when successive effects are not additive but rather multiply each other (as do amplification and absorption), we need additive units for convenience. Here are suggestions for simplifying nomenclature and avoiding confusion.

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LOGARITHMIC SCALES ARE useful in many fields. Sound power attenuation, musical pitch, star magnitudes, photographic exposure parameters, hydrogen ion concentration and optical densities are all expressed on logarithmic scales, but how different the usages appear! As each technology satisfied its need in an ad hoc manner, the systems took on various peculiar forms. As a result, the essential equivalence of the forms is not generally recognized and the concepts and terminology are usually treated as special cases.

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Opacity, for example

Typical of the applications is optical density, the logarithm to base ten of the ratio of incident flux to transmitted flux. Being a logarithm of a ratio of fluxes, this quantity is generally regarded as having no units. Nonetheless, in discussions involving optical density one often hears phrases such as "a difference of one tenth density unit." The correct phrase would appear to be, "a difference in density of one tenth," but this wording might suggest that the difference is one tenth of the original, lower or mean density. Thus there is clearly some advantage in the concept of a unit of density and some merit in a name for communication.

The ratio of fluxes to which we have referred is called "opacity" in photographic technology (though not in optics generally). The property it measures, which we can call "opaqueness," is a measurable physical quantity of which one can select an amount to be designated one unit. One could set aside a sample having an opaqueness arbitrarily designated as one unit. Such units would be additive, in the sense that two such layers would produce twice the optical effect of one, only if the effect were expressed in terms of a logarithm of the flux ratio rather than in terms of the flux ratio itself. This, of course, is the basic reason optical density is defined as it is. Thus, we see that the quantity has the physical realizability and additivity required of a unit.

A general rule

Any positive real ratio a/c can be expressed by two numbers, b and n, in the exponential form

$$a/c = b^n$$

Either b or n can be chosen arbitrarily, with certain obvious restrictions, but the choice of one determines the other. Regardless of logarithmic base

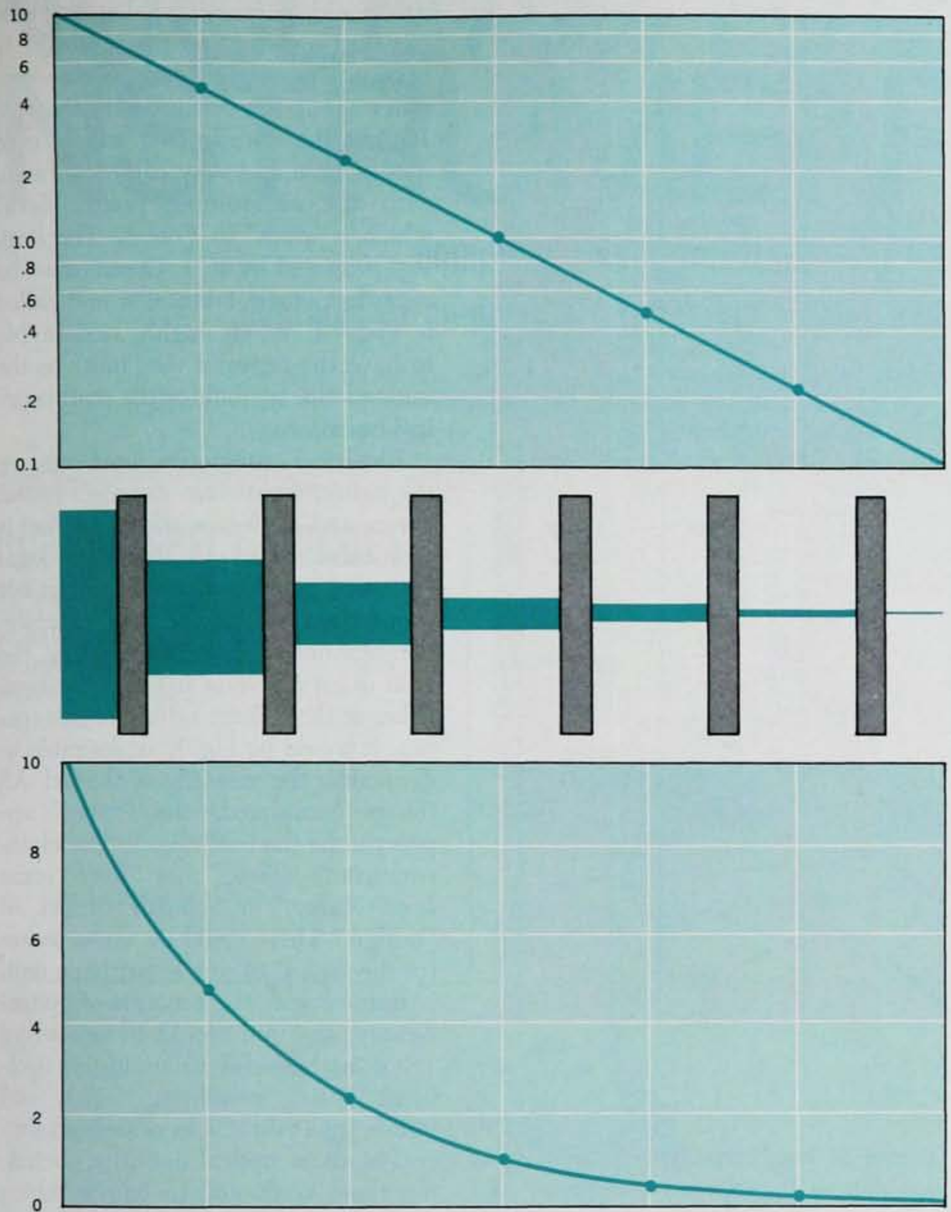
$$\log (a/c) = n \log b$$

By choosing the base b, we can write

$$\log_b (a/c) = n \log_b b$$

We can regard n to be the numerical measure and $\log_b b$, which is equal to 1, to be the unit of the quantity $\log_b (a/c)$ on a logarithmic scale. I propose to call this unit an "order to base b" (symbol: "ord_b"). If $\log_b (a/c) = n$, then a is n orders to base b greater than c. Taking one order as a unit, one can use the standard prefixes, such as deci-, centi-, and milli-, to form multiples and submultiples.

This use of the word "order" is by no means a totally new departure. Webster's Third New International



ADVANTAGE of logarithmic units is illustrated by plotting optical-absorption data on both log and linear scales.

Dictionary lists among the various meanings of "order" "general or approximate size, quantity, or level of magnitude or a figure indicative thereof." I have asked a number of scientists how they would quantitatively interpret the assertion, "The number of normal specimens is a couple of orders greater than the number of defective ones." This was generally taken to mean, "There are about 100 times as many normal specimens as there are defective ones." Thus the word "order" generally meant what would be called an "order to base ten," the base related to our number system. The present proposal is merely to generalize the usage to other bases. We would undoubtedly continue to use the term without the base modifier to mean "to base ten."

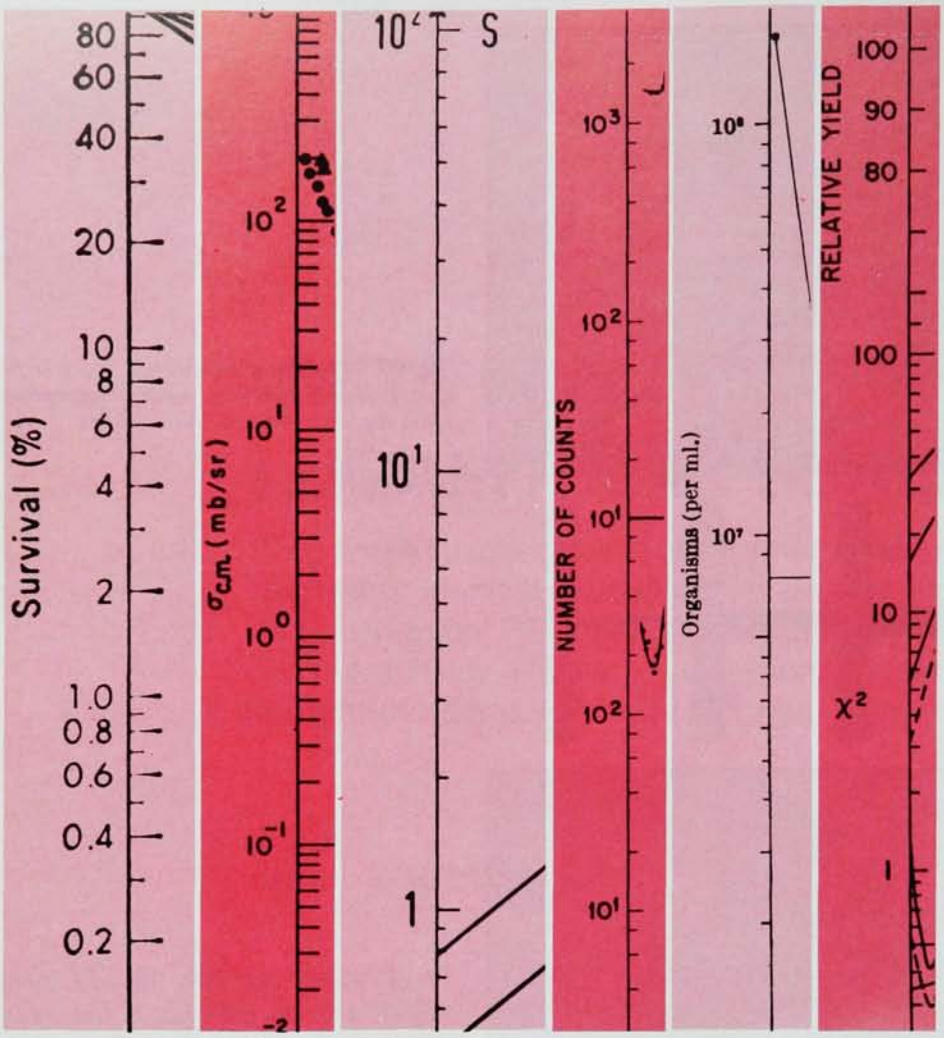
Existing systems

Armed with this definition, we may easily discuss various logarithmic systems. In acoustics and electronics, power ratios of attenuation or gain are treated logarithmically to base 10 and one order is called a "bel," (symbol "B") in honor of the acoustical engineer Alexander Graham Bell. The decibel (dB) is a common submultiple.[1]

Hipparchus and Ptolemy rated the magnitudes of stars on a scale ranging from 1 for the brightest to 6 for the faintest. John Herschel noted that a star of magnitude 1 was 100 times as bright as one of magnitude 6, which led Pogson to propose a logarithmic scale of stellar magnitude to base $100^{1/5}$, or approximately 2.512. The magnitude is the negative $\log_{2.512}$ of the brightness. One order is called a "magnitude." The name of the unit is the name of the quantity.

The tempered chromatic musical scale is a geometric progression of frequencies having a ratio of the 12th root of 2, this ratio being called a "half step." The tone 12 half steps above a given tone has a frequency twice that of the given tone. This scale is logarithmic to base 2, with duodecimal subdivision. The most common scale derived from this scale is the tempered major scale having a pattern of seven ratios involving half steps and whole steps, the latter being the sixth root of two. If we count the first tone and the last, there are eight in the major scale. On this somewhat artistic basis, the order to base 2 is called an octave, from the Latin "octava," an eighth. This unit is common in harmonic analysis in general.[2] In the absence of a standard symbol, I propose "oc."

Although photographic exposure parameters are quite generally regarded as being treated on scales that are logarithmic to base 2, the scales in actual use for film speeds and shutter markings include the numbers 500 and 1000 rather than 512 and 1024 and can be much more closely approximated by scales that are logarithmic to base $10^{3/10}$, which is 1.9953. . . . Subcommittee PH2-27 of the United States of America Standards Institute has recommended this scale for the exact values of exposure parameters, and it was used by Subcommittee PH2-22 in the revision of the United States of America Standard Photo-



graphic Exposure Guide. In this guide, one order to base $10^{3/10}$ is called a "step." No symbol was assigned. I propose "st."

The logarithms to the natural base e of voltage ratios and ratios of various other electrical and mechanical quantities are often used. An order on this scale is called a "neper" (symbol: "Np"), honoring John Napier, the inventor of logarithms. Incidentally, his $\log x$ may be expressed in terms of what are now called "natural" or "Naperian" logarithms as $10^7 \log_e (10^7/x)$.

Some authors[3] have suggested the name "brigg" (symbol: "Br") for what I have called one "order to base 10," honoring Henry Briggs who first published logarithms to base 10. This usage would be exactly analogous to the standard neper (Np) for one order to base e. The name "decade" is sometimes applied to one order to the base 10, but the decade is a widely used unit of time and may cause confusion with the submultiple prefix "deka" which is "deca" in French. The decilog, proposed by E. I. Green, must be regarded as one tenth of a unit called a "log." It seems highly undesirable to have the name of the "unit" be the same as the symbol for the mathematical operator.

Electrical engineers must reserve the bel for logarithms of power ratios. Errors and confusion arise if the bel is erroneously used to designate logarithms of quantities related to but not proportional to power. For example, the number of bels representing the gain of an amplifier is twice the logarithm of the voltage ratio. For this reason, it would be highly undesirable to generalize the meaning of the bel. All things considered, the "brigg" appears to be the best choice. The mathematician's name may have come from "brigg," a Scottish variant of "bridge." There could be worse names for the "span" of one logarithmic unit.

Returning to the example of optical density, the unit would be one brigg (Br), with useful submultiples decibrigg (dBr), centibrigg (cBr), and millibrigg (mBr). I have defined several kinds of optical densities, including those computed to base e which were called "natural densities."[4] These would have the unit neper (Np).

The standardization and general use of the terms proposed here should help clarify the usage of logarithmic scales and forestall needless proliferation of specialized terminology, units, and symbols.

The existing and proposed terms and symbols are given in the table.

Existing and Proposed Terms and Symbols

Physical quantity	*Base*	*Name of one order*	*Symbol*
Power attenuation or gain	10	bel	B
Stellar magnitude (brightness^{-1})	$100^{1/5}$	magnitude	
Musical pitch and other harmonic analysis (frequency)	2	octave	oc
Photographic exposure settings	$10^{3/10}$	step	st
Various electrical, acoustical, and mechanical (proposed for general use)	e	neper	Np
(proposed for general use)	10	brigg	Br
(proposed for general use)	b	order to base b	ord_b

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References

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2. A. G. Pikler, "Logarithmic Frequency Systems," J. Acoust. Soc. Am. **39**, 1102 (1966).
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4. C. S. McCamy, "Concepts, Terminology, and Notation for Optical Modulation," Photographic Science and Engineering **10**, 314 (1966). □