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AN OPERATIONAL INTERPRETATION OF

NONRELATIVISTIC QUANTUM MECHANICS

Idealized experiments show that a physical measurement involves three stages: initial state preparation, perturbation and examination for the probability that a particular final state is occupied.

WILLIS E. LAMB JR

WHAT IS QUANTUM MECHANICS? A remarkable feature of the 1968 conference of Nobel prize winners in physics at Lindau is that it was possible for me to ask such a question in the presence of two of the founders of quantum mechanics, Werner Heisenberg and P.A.M. Dirac, more than 30 years after the discovery, in a lecture attended by 400 students who had recently begun their study of the subject. Several answers to the question are possible. The only easy one is that quantum mechanics is a discipline that provides a wonderful set of rules for calculating physical properties of matter. For such simple systems as hydrogen and helium atoms the calculated energy levels agree with experiment to fantastic accuracy. In more complicated cases the computations are difficult and the accuracy is lower, but it is reasonable to believe, in principle at least, that the theory would be adequate if only the calculational problems could be overcome.

A discussion of the interpretation of quantum mechanics on any level beyond this almost inevitably becomes rather vague. The major difficulty involves the concept of "measurement," which in quantum mechanics means determining the value of a physical observable for a dynamical system with as much precision as is possible.

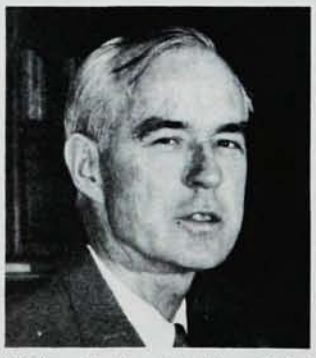
I have taught graduate courses in quantum mechanics for over 20 years at Columbia, Stanford, Oxford and Yale, and for almost all of them have dealt with measurement in the following manner. On beginning the lectures I told the students, "You must first learn the rules of calculation in quantum mechanics, and then I will tell you about the theory of measurement and discuss the meaning of the subject." Almost invariably, the time allotted to the course ran out before I had to fulfill my promise.

The truth was that I found most available discussions of quantum-mechanical measurement either too vague, or too formal and unphysical. However, as the years went by, I gradually put together for myself an interpretation of nonrelativistic quantum mechanics that stresses an operational point of view. My attitude toward such problems has no doubt been influenced by contact with some research in experimental physics in which atomic states are "manipulated" by microwave and radiofrequency fields. In discussion of the measurement of some dynamical variable of a physical system I want to know exactly what apparatus is necessary for the task and how to use it, at least in principle. I am not satisfied with hand waving or with a formal logical scheme involving black boxes.

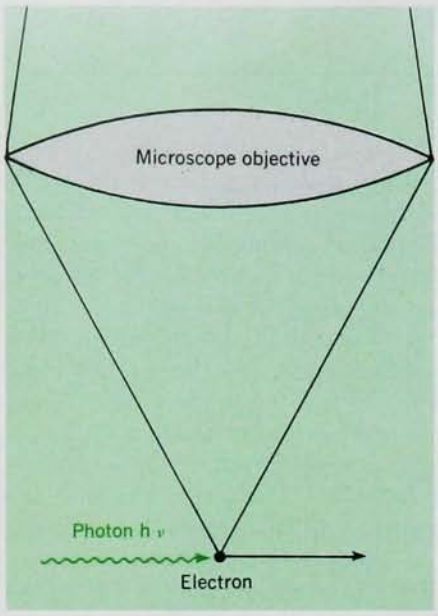
Mark Twain once said about meteorology: "Everybody talks about the weather, but nobody does anything about it." A similar statement could be made about the subject of measurement in quantum mechanics. Whereas no one, in fact, could actually carry out any of the experimental procedures I will be discussing, they represent an ideal limit that is worth considering. Granting reasonable premises, one knows, as far as possible, at all times, exactly what one is doing to the physical system under consideration. If, in practice, one is less than certain about what he is doing to the system, then it appears obvious to me that his knowledge of its future state will be reduced. I believe, in fact, that the measurements I will describe are the most precise ones possible. They involve the use of idealized apparatus that may not be realizable in practice. Actual measurements are less than ideal. In almost all physical experiments where one says that something is "measured" there is at most only a very bad quantum-mechanical measurement in the technical sense of the term. This is certainly true of the scattering experiments of atomic and nuclear physics. For example, the famous "gamma-ray microscope" (figure 1) used by Heisenberg[1] for qualitative discussions of his uncertainty relations does not provide a measurement of position but a scattering experiment. In my opinion, a quantum-mechanical theory of imperfect measurements, in which one settles for less than the best imaginable experimental control, has still to be given. It will probably make use of the theory of random processes.

Spinless electrons in a field

Let us consider that the physical system under study is a nonrelativistic



Willis E. Lamb Jr's Nobel Prize was awarded in 1955 for the discovery of what everyone now knows as the "Lamb shift" in the fine structure of the hydrogen spectrum. He served on the physics faculties of Columbia, Stanford and Oxford Universities before taking up his present position as Henry Ford II professor of physics at Yale in 1962.



GAMMA-RAY MICROSCOPE postulated by Heisenberg in his discussions of the uncertainty relations. The electron cannot be observed unless it interacts with a photon, and then its recoil makes accurate simultaneous position and momentum measurements impossible. –FIG. 1

spinless electron of mass m moving in a known conservative force field $V(x)$. Its Hamiltonian operator is the sum of a kinetic and potential energy

$$H = T + V$$

with

$$T = -\frac{\hbar^2}{2m}\nabla^2, \qquad V = V(x)$$

Associated with the electron is a wave function $\psi(x,t)$ that obeys the Schrödinger wave equation

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = (T + V)\,\psi(x,t)$$

This differential equation arises from an attempt to generalize classical mechanics by introducing a wave field $\psi(x,t)$ somehow describing the electron. According to the basic assumption of quantum mechanics

$$dW(x,t) = |\,\psi(x,t)\,|^2\,dx$$

gives the probability at time t that the electron is between x and $x + dx$. This probability statement refers not to a single system but to an ensemble of systems all of which are described by the same wave function $\psi(x,t)$. In further development of the theory, the probability assumption is generalized as follows: Consider a complete set of commuting Hermitian operators (observables) $F(x,p)$ with eigenvalues F_f and eigenfunctions $\psi_f(x)$ obeying the equation

$$F(x,p)\;\phi_f(x) = F_f\phi_f(x)$$

The functions $\phi_f(x)$ form a complete set of orthonormal expansion functions for the problem so that we may expand the wave function in the form

$$\psi(x,t) = \Sigma_f C_f(t)\;\phi_f(x)$$

It then follows plausibly that the absolute square of the expansion coefficient $C_f(t)$ is the probability

$$W_f = |C_f|^2$$

that the dynamical variables F "have" the values F_f for the state $\psi(x,t)$. Note that I am being rather vague about how the values F_f are to be determined. The probabilistic assumption carries with it no recipe for its experimental verification or rejection. It is also not clear whether all Hermitian operators F, or only some specially favored ones, can be measured in the sense implied by the probability assumption.

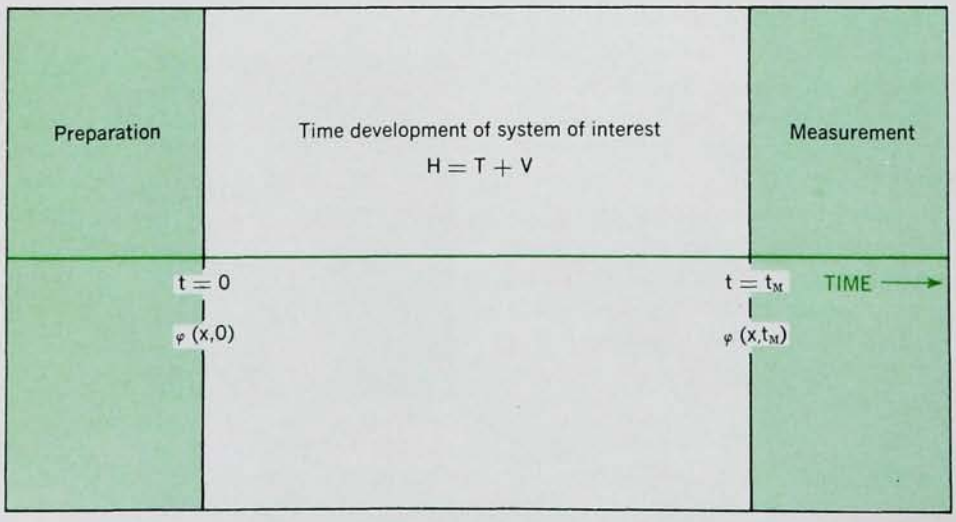
Classical state preparation

Before indicating how measurements are to be made, I take up an important related question, the *preparation* of a state $\psi(x,0)$ at time $t = 0$. (Although some authors confuse *preparation of a state* and *measurement*, these concepts are logically and physically very different.) It is clear that the time-dependent Schrödinger equation will determine the future wave function $\psi(x,t)$ if its initial form $\psi(x,0)$ is known. For this to be useful we should have some way to start the system off with the initial wave function so that we can derive some physical information from its calculable value $\psi(x,t_M)$ at a time t_M of measurement. This is indicated schematically in figure 2. Just before $t = 0$ certain procedures called preparation of the initial state are carried out. From $t = 0$ until $t = t_M$ the system's wave function develops according to its Schrödinger wave equation. Just after $t = t_M$, a set of procedures constituting the measurement of some dynamical variable $F(x,p)$ are carried out on the system.

The first question to answer is: Can one give an experimental procedure for bringing the electron initially into a state having an arbitrarily assigned wave function $\psi(x,0)$? The answer is "Yes," provided that certain assumptions are made. Consider the corresponding problem in classical mechanics where the particle has the Hamiltonian $H(x,p)$. The initial state of a particle is fully specified by its classical coördinate $x(0)$ and momentum $p(0)$, and the future state $x(t)$, $p(t)$ is determined uniquely by integrating the Hamiltonian equations of motion

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x}$$

subject to the initial conditions at $t = 0$. In classical mechanics one would ordinarily consider the preparation of the initial state as trivial. At $t = 0$ one simply places the particle at $x(0)$ and gives it a momentum $p(0)$. This can be done entirely within the bounds of classical mechanics. We set up a potential well $U_1(x)$ having a minimum at $x = x(0)$. Then we somehow catch a particle in the potential well and allow it to settle down to rest at the minimum with the assistance of a little friction. It is then located at



STATE PREPARATION AND MEASUREMENT. The wave function of a system develops from time $t = 0$ to time $t = t_M$ according to Schrodinger's time-dependent wave equation. The process of "preparation" of the initial state occurs before $t = 0$. A measurement of some physical quantity is begun at time t_M. –FIG. 2

$x(0)$ at $t = 0$. We next give the particle an impulsive force derived from a potential $U_2(x)\ \delta(t)$ where $\delta(t)$ is the Dirac delta function, sharply peaked at $t = 0$. This force can bring the particle momentum to $p(0)$ without changing its position appreciably. Just afterwards, the potential $V(x)$ corresponding to the problem of interest is turned on. As we want to be able to do this for arbitrary values of $x(0)$, $p(0)$ it seems reasonable to assume that we can apply arbitrary potentials of the form $U(x,t)$ to the system and that we can introduce a small amount of frictional force when needed.

Quantum-mechanical preparation

We now turn to the problem of state preparation in quantum mechanics. It may come as a surprise that this requires, at least in principle, exactly the same experimental facilities as the classical case needed—and no more.

Suppose we want to prepare a state $\psi(x,0)$. If this happens to be known to us as the lowest energy eigenfunction for a certain potential, all we have to do is to set up this potential, somehow catch an electron in it, and wait for radiative damping to bring the particle to the ground state. Then we suddenly turn off $U_1(x)$ and turn on the $V(x)$ for the problem of interest. (According to sudden-perturbation theory[2] a wave function is not immediately changed by a sudden change in potential if no singular time dependence is involved.) The wave function $\psi(x,t)$ for later times would then be determined by the time-dependent Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = H\psi = (T+V)\psi$$

for the system of interest. There is, of course, the problem of recognition that we have caught an electron in $U_1(x)$, but I regard this as trivial in principle and do not discuss it here.

If the desired initial state is not the known lowest-energy eigenfunction of some potential $U_1(x)$ a number of procedures are possible. We may look for a potential $U_1(x)$ in which $\psi(x,0)$ is an eigenstate of some energy E, if not the lowest one. Because $\psi(x,0)$ obeys the stationary-state wave equation

$$[T + U_1(x)]\,\psi(x,0) = E\,\psi(x,0)$$

we can solve for the required potential $U_1(x)$ and find explicitly that

$$U_1(x) = E - [\psi(x,0)]^{-1}\,T\,\psi(x,0) \quad (1)$$

If this potential $U_1(x)$ is experimentally available, we catch the electron in it. The wave function will then be some linear combination of the eigenstates of the Hamiltonian $T + U_1$, one of which is the desired $\psi(x,0)$. There are several possibilities here. One is to carry out a state-selection process of the Stern–Gerlach type to collect the atom if its energy eigenvalue is E. If we get some other eigenvalue, we reject the atom and repeat the procedure until we get the desired one.

Complications

Three difficulties might occur: The potential $U_1(x)$ given by equation 1 might be singular if $\psi(x,0)$ has nodes; the energy spectrum might not be discrete so that state selection would be difficult, or the potential $U_1(x)$ might be a complex function because $\psi(x,0)$ is not a real function. These troubles can either be ignored, or overcome by methods such as the one I will discuss here only for the case that the function $U_1(x)$ given by equation 1 turns out to be complex.

In that case, we write the desired initial wave function in the form

$$\psi(x,0) = R(x)e^{iS(x)}$$

where $R(x)$ and $S(x)$ are real functions of position. Then we use a real potential

$$U_1(x) = E - [R(x)]^{-1}\,T\,R(x) \quad (2)$$

instead of equation 1 and prepare a state with the real wave function $R(x)$ in the manner previously described. We next apply a pulse potential

$$U_2(x,t) = -\hbar\,S(x)\,\delta(t) \quad (3)$$

that is designed to convert the wave function from $R(x)$ to $\psi(x,0)$. During the pulse, Schrödinger's wave equation can be adequately approximated by

$$-\frac{\hbar}{i}\frac{\psi}{t} = -\hbar\,S(x)\,\delta(t)\,\psi(x,t)$$

or

$$\frac{\partial\psi(x,t)}{\partial t} = i\,S(x)\,\delta(t)\,\psi(x,t)$$

The solution of this differential equation is easily found. Just after the pulse the wave function is

$$\psi(x,0_+) = R(x)e^{iS(x)}$$

if initially we had the wave function $R(x)$. In this way, by first applying the potential $U_1(x)$ of equation 2, then $U_2(x,t)$ of equation 3 and finally $V(x)$, we can prepare any desired initial state of the system of interest. Note the close parallel with the classical situation.

Spreading wave packets

We might illustrate this procedure with a simple example. Consider a free particle whose Hamiltonian is

$$H = \frac{p^2}{2m}$$

and that has a real Gaussian wave function at $t = 0$,

$$\psi(x,0) = (2\pi)^{-1/4}(\triangle x_0)^{-1/2} \exp\left\{-\frac{1}{4}\frac{x^2}{(\triangle x_0)^2}\right\}$$

The corresponding probability density

$$\rho(x,0) = |\psi(x,0)|^2 = (2\pi)^{-1/2}(\triangle x_0)^{-1} \exp\left\{-\frac{1}{2}\frac{x^2}{(\triangle x_0)^2}\right\}$$

is also Gaussian and has a width $[\langle x^2\rangle - \langle x\rangle^2]^{1/2} = \triangle x_0$. According to the time-dependent wave equation, the wave function after elapse of a time T is the complex Gaussian expression[3]

$$\psi(x,T) = (2\pi)^{-1/4} [\triangle x_0 + \frac{1}{2}(i\hbar T)/(m\triangle x_0)]^{-1/2} \times \exp\left\{\frac{-x^2/4}{[(\triangle x_0)^2 + (1/2)\,(i\hbar T)/m]}\right\}$$

for which the probability density is

$$\rho(x,T) = |\psi(x,T)|^2 = (2\pi)^{-1/2}\,[(\triangle x_0)^2 + \frac{1}{4}\,(hT)^2/(m\triangle x_0)^2]^{-1/2} \times \exp\left\{\frac{-x^2/2}{[(\triangle x_0)^2 + (1/4)\,(hT)^2/(m\triangle x_0)^2]}\right\}$$

During the time T the wave packet has spread, remaining Gaussian, but its width parameter $\triangle x$ increases with time according to the equation

$$(\triangle x)^2 = (\triangle x_0)^2 + \frac{\hbar^2 T^2}{4m^2(\triangle x_0)^2}$$

This has an obvious interpretation in terms of Heisenberg's uncertainty relationship

$$\triangle x\,\triangle p \geq \frac{1}{2}\hbar$$

The initial spread $\triangle x_0$ in x implies an uncertainty in p, and this after a time T implies an additional spread in x equal to $\hbar T/(2m\triangle x_0)$, which has to be added in quadrature to the original $\triangle x_0$.

The above wave packet spreads for $t > 0$. If we could start at time T with a wave packet $\psi(x,T)$ and reserve the arrow of time, the width of the probability distribution would shrink from $\triangle x$ at $t = T$ to $\triangle x_0$ at $t = 0$, whereas it would spread again for $t < 0$. Unfortunately, no one knows how to reverse the flow of time, although many theoretical papers ignore the difficulty. However, if we could start at time $t =$

0 with a wave packet having the initial form

$$\psi(x,-T) \equiv \psi(x,T)^*,\ T>0$$

then the probability distribution would shrink from a width $\triangle x$ at time $t=0$ to $\triangle x_0$ at the later time $t=T$; then spreading would occur. There is no limit (in nonrelativistic quantum mechanics) to the sharpness $\triangle x_0$ of the wave packet that could thereby be achieved at time T. We must merely start with the proper complex wave packet $\psi(x,-T)$ with an enormous width parameter $\triangle x$. The question then arises: Can this wave packet be realized experimentally, and if so, how? Our general recipe for preparation of a state gives the answer. The complex wave function $\psi(x,-T)$ has to be written in the form

$$\psi(x,-T) = R(x)e^{iS(x)}$$

After a simple calculation we find that

$$R(x) = (2\pi)^{-1/4}\,(\triangle x)^{-1/2} \exp\left\{-\frac{1}{4}\frac{x^2}{(\triangle x)^2}\right\}$$

and apart from an additive constant

$$S(x) = -\frac{1}{8}\frac{\hbar T}{m(\triangle x_0)^2}\frac{x^2}{(\triangle x)^2}$$

The function $R(x)$ is the ground-state wave function for a simple harmonic-oscillator potential

$$U_1(x) = \frac{1}{2}\frac{h^2}{m(\triangle x)^2}\frac{x^2}{(\triangle x)^2}$$

and if $\triangle x$ is to be very large, the related spring constant must be very small. The pulse potential

$$U_2(x,t) = -\hbar\, S(x)\,\delta(t)$$

that should then be applied is also parabolic in its space dependence.

This simple example illustrating a much more general procedure admits of a simple interpretation. The shrinking of the wave packet is accomplished when the broad Gaussian wave function $R(x)$ is acted on by a pulse potential that delivers to the electron, whatever its coördinate x might be, just the impulse required to bring it very near $x=0$ after the passage of a time T. The procedure has certain similarities to the focusing of a beam of light by a lens.

We have now shown how to prepare an arbitrary initial state. Subsequently, this state evolves according to the time-dependent Schrödinger equation for the system of interest. If the system is isolated, its Hamiltonian is of the form $H=T+V$. We may even apply external perturbations described by known time-dependent conservative force fields, and the wave function will still develop in a causal manner. If the perturbation has some uncertain or random element, the system will no longer be describable by a wave function, but by a statistical mixture of wave functions. The density matrix of John von Neumann,[4] Lev Landau[5] and Dirac[6] provides a convenient description for such situations. However, we consider only the pure case here.

Dynamical variables

Suppose at some time t_M that we wish to make a measurement of a real dynamical variable $F(x,p)$ for the system then described by the wave function $\psi(x,t_M)$. How is this to be done in an operational manner? Is it, in fact, possible for any Hermitian operator $F(x,p)$, or just for some specially favored ones? Dirac[7] has indicated that it should be possible to measure "observables," which he defines as real dynamical variables whose eigenstates form a complete set; however Wolfgang Pauli[8] and others have doubted that the question can be decided within the framework of nonrelativistic quantum mechanics.

If $F(x,p)$ were in the form of an experimentally realizable Hamiltonian operator, one could manage to measure F very nicely as follows. Just at time t_M one would suddenly switch off $V(x)$ and instead subject the electron to the Hamiltonian $F(x,p)$. Let the eigenfunctions of $F(x,p)$ be denoted by $\phi_f(x)$ and the corresponding eigenvalues by F_f. For simplicity, we assume that any degeneracy has been lifted so that there are no repeated eigenvalues. We may then expand the wave function $\psi(x,t_M)$ in a series of the $\phi_f(x)$ as

$$\psi(x,t_M) = \sum_f C_f\,\phi_f(x)$$

At some later time $t > t_M$ the wave function would be

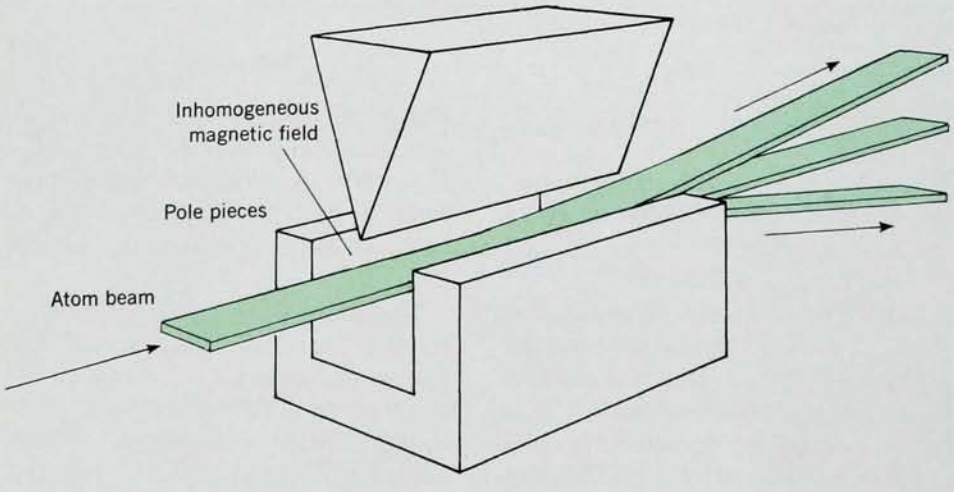
$$\psi(x,t) = \sum_f C_f\,\phi_f(x) \exp\left\{-(i/h)F_f(t-t_M)\right\}$$

The probability W_f of finding the system in state f, that is, having a value of the operator F equal to F_f, is given by the absolute square of the expansion coefficient C_f,

$$W_f = |C_f|^2$$

and is independent of t. One then has a long time available to decide how to determine the probability distribution over the states f. The method usually contemplated is to send the atoms through some kind of Stern–Gerlach apparatus (figure 3). This has been often discussed as in monographs and textbooks of Pauli,[9] David Bohm,[10] Kurt Gottfried[11] and others, and I do not have much to add to this part of the story. At some stage in the measurement process the wave function is converted into a mixture through the coupling of the quantum-mechanical system to a quasi-classical system, which in effect has a random interaction with the system of interest. A further stage, essentially not quantum mechanical, involves recognition that one has caught the atom, as on a photographic plate, and is required to complete the measurement process.

We have assumed here that the Hermitian operator $F(x,p)$ could be regarded as a possible Hamiltonian op-



STERN–GERLACH APPARATUS. An inhomogeneous magnetic field between specially shaped pole pieces produces forces on atoms passing through; the forces are proportional to magnetic quantum number; so different states of the atoms can be separated. In some cases inhomogeneous electric fields could be used instead of these inhomogeneous magnetic fields. **–FIG. 3**

erator for the electron. In nonrelativistic quantum mechanics, however, in the absence of a magnetic field, all Hamiltonians are of the form $H = T + V$, where the potential V may depend on the coördinate x and time t, but not on momentum p. An easy way out is to ignore this difficulty, and to claim that the laws of quantum mechanics must prevail no matter what the Hamiltonian may be. This theoretical possibility will not suffice if we are to have an operationally possible procedure for the measurement of $F(x,p)$.

Even if $F(x,p)$ is not a Hamiltonian operator, it may still be possible to give a prescription for measuring it. We first consider some special cases of great importance. The momentum operator p is not in Hamiltonian form, but $p^2/2m$ is the Hamiltonian for a free particle. In his 1929 Chicago lectures Heisenberg[12] mentioned that the momentum of an atomic electron could be measured by suddenly turning off the atomic potential and letting the electron wave function spread. The distribution of times of arrival for electrons at a distant detector would give the desired momentum distribution.

Position coordinates

Another operator of basic importance is the coördinate x, which is also clearly not in Hamiltonian form. Nevertheless it can be measured by a simple procedure. Suppose x is to be measured at time t_M for a particle whose wave function is $\psi(x,t_M)$. That means that we want the probability distribution for finding the particle in the range x_M, $x_M + dx_M$. As usual, we have to study an ensemble of a lot of similarly prepared systems. We suddenly turn on a very strong but short-range attractive potential centered about the point x_M. Let this potential have only one bound state with energy E_1 and wave function $u_1(x - x_M)$ as shown in figure 4. The original potential $V(x)$ is simultaneously turned off. For times t later than t_M the wave function $\psi(x,t)$ can be expanded in terms of the eigenfunctions $u_n(x)$ and energy E_n, for the short-range potential

$$\psi(x,t) = \sum_n C_n u_n(x - x_M) \exp\left\{-(i/h)E_n(t - t_M)\right\}$$

where the sum has one discrete term $n = 1$ and also implies an integration over the continuous spectrum of unbound states. After a fairly short time, all parts of the wave function will spread out and become negligible near x_M except the term in $u_1(x - x_M)$. The probability amplitude C_1 is given by

$$C_1 = \int u_1(x - x_M)^* \psi(x,t_M)\, dx$$

and the time-independent probability for catching the electron in the potential is

$$W_1 = |C_1|^2 \qquad (4)$$

If $u_1(x - x_M)$ has a short range compared with the distance in which $\psi(x, t_M)$ varies appreciably, equation 4 becomes approximately

$$W_1 \propto |\psi(x_M,t_M)|^2$$

When this is normalized, we find that

$$dW\ (x_M) = |\psi(x_M,t_M)|^2\, dx_M$$

in agreement with the original probability assumption of quantum mechanics. The method is similar to one that might be used to determine the distribution of flies in a room. One would quickly clasp one's fingers about a small region x_M and find out by subsequent operations whether one had caught a fly or not. Then the process would be repeated many times for similarly prepared rooms to build up a probability distribution.

Measuring Hermitian operators

In general, the non-Hamiltonian operator $F(x,p)$ will not permit itself to be measured by such simple tricks, and one might begin to doubt whether all Hermitian operators can be measured. However, the following recipe has been developed following suggestions by Yakir Aharonov of Yeshiva University. It gives a close formal, if not very physical, correspondence between the operator $F(x,p)$ and the experimental procedures to be followed. We first have to find a set of operators $G(x,p)$ that together with $F(x,p)$ form a complete set of commuting observables. This should not be hard to do in any specific case, although no general prescription is available. The simultaneous eigenfunctions of F and G should form a complete set of expansion functions spanning the Hilbert space of the problem of interest. They obey equations of the form

$$\begin{aligned} F(x,p)\ \phi_{fg}(x) &= F_f \phi_{fg}(x) \\ G(x,p)\ \phi_{fg}(x) &= G_g \phi_{fg}(x) \end{aligned} \qquad (5)$$

where the subscripts f, g label eigenfunctions and eigenvalues. (For simplicity, only one equation of type (5) is written.) The wave function $\psi(x,t_M)$ on which the measurement of F is to be made can be expanded in the form

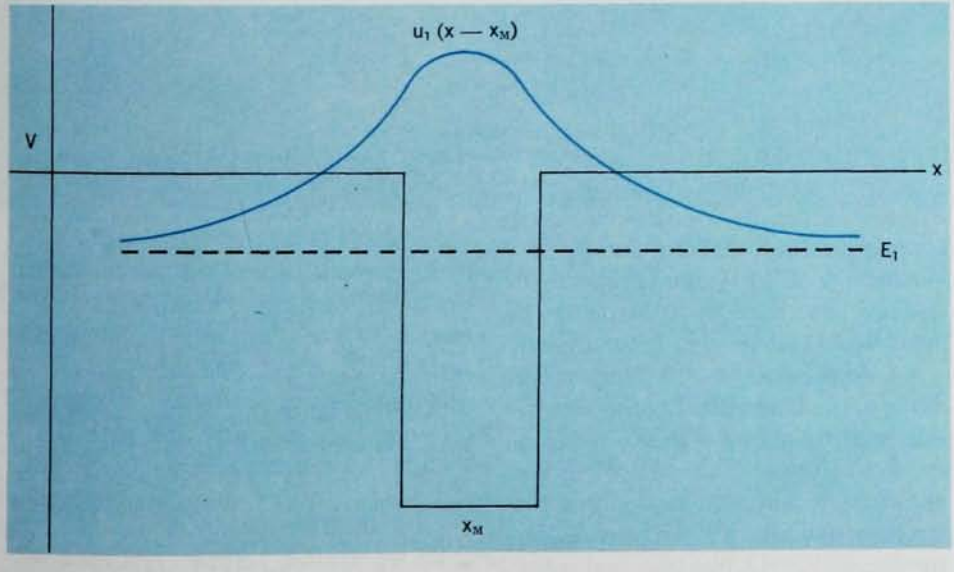
$$\psi(x,t_M) = \sum_f \sum_g C_{fg} \phi_{fg}(x)$$

where the expansion coefficients are given by

$$C_{fg} = \int \phi_{fg}(x)^* \psi(x,t_M)\, dx \qquad (6)$$

The absolute square of C_{fg} gives the probability that F has the value F_f and G has the value G_g. If we care only about F, we must carry out a sum over g.

Let us indicate an experimental procedure that can give the value of the



MEASUREMENT OF POSITION means determining the probability distribution for catching a particle in a short-range potential trap. This potential is centered at x_M and has one bound state $u_1(x - x_M)$ of energy E_1. The wave function describing the electron, $\psi(x,t_M)$, spreads out when the short-range potential is turned on and the original system potential $V(x)$ is turned off. All parts of the wave function quickly become negligible except for the term in $u_1(x - x_M)$, which is shown in color. —FIG. 4

probability distribution

$$W_{fg} = |C_{fg}|^2$$

Suppose that the eigenfunction ϕ_{fg} is written in terms of certain real functions R and S as

$$\phi_{fg}(x) = R_{fg}(x) \exp\left\{ -iS_{fg}(x) \right\}$$

where both R and S depend on the indices f and g. The first step in the measurement then involves converting the wave function $\psi(x,t_M)$ of the system into a new wave function $\psi_S(x)$ given by

$$\psi_S(x) = \psi(x,t_M) \exp\left\{ iS_{fg}(x) \right\} \qquad (7)$$

by applying a pulse potential

$$U_2(x,t) = -\hbar S_{fg}(x)\, \delta(t - t_M)$$

to the system. Then the probability amplitude C_{fg} of equation 6 has the value

$$C_{fg} = \int R_{fg}(x)\, \psi_S(x)\, dx$$

which is just the overlap integral between the new wave function $\psi_S(x)$ of the system and the real normalized wave function R_{fg}. From the earlier discussion of state preparation we know how to find a potential $U_{fg}(x)$ in which $R_{fg}(x)$ is an eigenfunction of energy E. This potential is given by

$$U_{fg}(x) = E - [R_{fg}(x)]^{-1}\, T\, R_{fg}(x)$$

Hence, the second stage of the measurement process involves the sudden application of the potential $U_{fg}(x)$ and removal of the potential $V(x)$ from the system with a wave function given by equation 7. The next task is to find the probability that the particle is in the state of energy E. This could be accomplished by the usual kind of Stern–Gerlach procedure. In this manner, we find the desired probability that the operators F, G have values F_f, G_g for the state $\psi(x,t_M)$ of the system of interest. It will be noted that the measurement problem is now more complicated than for Hamiltonian operators, as the potentials $U(x,t)$ to be applied depend on the values of f and g, and hence a series of measurements has to be made for each set of f, g values.

Limitations

Some concluding comments are in order.

• We have assumed that all classically describable potentials $U(x,t)$ are available to us experimentally. This is quite similar and closely related to an assumption made by Niels Bohr and Leon Rosenfeld[13] in their discussion of the measurement of electromagnetic fields, that test bodies of very great mass and charge density were available to them, whose quantum-mechanical fluctuations of position and momentum could be neglected.

• Our discussion of measurement considers the observation of any one dynamical variable, A. Instead, we could measure another one, B, for the same state $\psi(x,t_M)$. For each operator we could determine the dispersion measure $\triangle A$ or $\triangle B$. The product of the fluctuation measures would obey the Robertson[14]–Schrödinger[15] generalization

$$\triangle A \cdot \triangle B \geq 1/2|<[A,B]>|$$

of Heisenberg's uncertainty relations. It should be noted, however, that such uncertainty relations do not refer to measurements, whether simultaneous or not, of a *pair* of observables. In special cases, involving commutability, it might be possible to measure first one observable and then another, but in general the thoroughgoing measurement of the first observable will so disrupt phase relations that it will serve no physical purpose to subsequently measure a second observable on the resulting mixture.

To measure simultaneously two noncommuting observables A and B (for example, x and p) one would have to find a potential $U(x,t)$ that was determined by both A and B. In general, I do not believe that this can be done in such a way that the desired information emerges from the measurement. One could, of course, form a single Hermitian operator out of the two Hermitian operators A and B. Some examples are $(AB + BA)$, $-i(AB - BA)$, $A^2B + BA^2$, ABA, etc. Any one of these Hermitian operators could be measured, as already indicated, but this would not be the desired simultaneous measurement of A and B.

• It is possible to extend the methods outlined so that measurements on many-body systems can be made.

• I do not see how to apply procedures of the kind outlined above to the relativistic quantum domain, or to field theory. In the absence of such generalizations, it may well be doubted that the story that I have given provides any significant insight into the real meaning of quantum mechanics. However, it is true that almost all expositions of quantum mechanics make use of the fictional notion that some kinds of measurements are possible. I have described certain experimental procedures for making them. There may be other ways. If they cannot be made in some fashion, either as I have suggested, or otherwise, then it appears that our understanding of the meaning of the quantum theory is correspondingly diminished, and it is only likely to be increased when a better theory of measurement for the more general relativistic and field-theoretic cases can be given. Of course, it may be that a system of rules for calculation can exist despite the absence of an operational interpretation of the kind I have attempted. For the teaching of quantum mechanics now, it is certainly a convenient fiction to pretend that the usual textbook assumptions about measurement have a meaning, even if from an operational point of view they do not. The mathematical formulation of quantum mechanics by Dirac beautifully matches the assumed notion of measurability. However, there is clearly much more for us to learn.

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