Nonlinear Problems In Physics

Interactions of matter and fields are generally nonlinear, so that nonlinear problems play a central role in physics. In fact because nonlinearity is so basic to nature, it is possible that even a theory as fundamentally linear as quantum theory may ultimately have to be replaced by a nonlinear one.

by Werner Heisenberg

NONLINEAR PROBLEMS in physics are difficult to discuss for several reasons. First, what are nonlinear problems? Practically every problem in theoretical physics is governed by nonlinear mathematical equations, except perhaps quantum theory, and even in quantum theory it is a rather controversial question whether it will finally be a linear or nonlinear theory. Therefore by far the largest part of theoretical physics is devoted to nonlinear problems.

Besides, it has been argued that every nonlinear problem is really individual; that is, it requires individual methods, usually very complicated and difficult methods, and it is rather improbable that one can learn from one nonlinear problem to solve another nonlinear problem.

Finally I have to emphasize that I am certainly not an expert in the field of nonlinear problems of mathematics or physics; but I have come across a few nonlinear problems on my way through physics, so I at least know some of the horrible difficulties and troubles which one meets in these problems.

In this situation I feel that the only thing I can do is not give a general survey of the field but rather tell about some of these problems that have passed my way in physics and see whether there are common features. In fact I do think that all these problems have some features in common—difficulties that are similar in different problems and also methods that one can use to solve the difficulties. So I hope you will allow me to go briefly through some of these problems and to see what are the common features.

I would like to start with a brief historical remark and to say a few words about point mechanics, and only then I will examine general problems of continuous media, and finally I will discuss to what extent quantum theory can be considered a linear or nonlinear theory.

Historical remarks

The course of history in physics usually proceeds from simple problems to more complicated problems. For the questions discussed here the simplest problem can be considered to be the solution of a homogeneous linear equation; next simplest is the solution of an inhomogeneous linear equation; and, finally, much more complicated is the solution of a nonlinear operation. Actually mathematical physics started 300 years ago with the law of inertia, which may be considered to be the solution of the homogeneous linear equation $d^2x/dt^2 = 0$ where x is the coördinate and t is the time. In the laws of free fall of Galileo, we find that he actually had solved an inhomogeneous linear equation, the force being the inhomogeneous term. Finally, Newton formulated the nonlinear problem because the equations of motion in Newton's mechanics definitely are nonlinear equations.

Point mechanics

Of course Newton was not in a position to find general solutions of nonlinear equations; so from the very be-

The author is famous for his contributions to the development of quantum theory and for his recognition of the fundamental importance of the uncertainty principle that now bears his name. He received the Nobel Prize for this work in 1932. Since 1941 he has been director of the Max Planck Institute for Physics and Astrophysics, Munich. This article is based on a talk he gave last summer at The International School of Nonlinear Mathematics and Physics, Munich, directed by Norman J. Zabusky, Bell Telephone Laboratories, Whippany, N.J., and Martin D. Kruskal, Princeton University, Princeton, N.J.

ginning he had to think about possible simplifications. Therefore, I might start by mentioning some of the most important simplifications that can be used, not only in the nonlinear problems of point mechanics, but actually in all nonlinear problems with which we have to deal.

Symmetry. There are mainly two different types of simplification that are important. The first comes from the symmetry of the problem. Physicists have learned from mathematicians that the symmetry of a problem as a rule produces a conservation law. All the conservation laws that we know in physics—conservation of energy, momentum, angular momentum etc.—rest upon fundamental symmetries in the underlying natural law.

For instance, Newton's equations of motion are invariant under the Galilean group, which is a continuous group of ten parameters. Therefore, one has ten conservation laws and these conservation laws can be used for the simplification of the problem. When Newton was able to solve the two-body problem in astronomy, it was by the use of symmetry. With two bodies there are 12 degrees of freedom because every body has 3 coördinates and 3 momenta; and of these 12 degrees of freedom, he could actually eliminate 10 by means of conservation laws, and the rest then was rather simple. Similar simplifications can be used in most problems.

Linearization. The next kind of simplification is linearization of the problem. This method has been very widely used and has actually been very important for the discussion of nonlinear problems. Let us assume that one has more or less by chance found simple solutions for the problem concerned, very special solutions—for instance, static solutions in which all the bodies are at rest, or as in hydrodynamics, stationary solutions in which the motion does not change with time; that is, the velocities at least are constant. In all these cases, one can study general solutions of the nonlinear problem in the neighborhood of the special solution. That is more or less the main object of perturbation theory. For small perturbations around a known motion one has linear equations to solve. Thereby one comes back to simpler mathematical problems for which one may be able to find the complete system of solutions. Such solutions can be studied for the whole time interval from minus infinity to plus infinity, and the solutions may be characterized by their asymptotic behavior in space and time.

If the special solution of the original system is a static or a stationary solution, then we have the special advantage that the coefficients in these linear equations of perturbation theory are constant in time, and it will be possible to look for solutions that behave like an exponential function of time, either $e^{i\omega t}$ or $e^{\alpha t}$.

If such a system of solutions has been found, one can at once answer questions of stability or instability. If only periodical solutions exist or only solutions that decrease in time, the system is stable—a perturbed solution starting at a given time will in the course of time not deviate strongly from the original simple solution. If, however, there are solutions of the linear equation that increase exponentially with time, the solution is not stable. This method of simplification can be applied even if the original solution is not a static solution. Many papers have been written on general perturbation theory and much insight has been gained by such methods.

Unpredictability. Beyond this point one comes very soon into the real difficulties of the nonlinear problems and I might first mention the difficulties in the case of point mechanics. For instance, consider the well known problem of three bodies in astronomy or the more complicated problem of our planetary system. There one has of course applied perturbation theory, but one has very soon found that it is extremely diffi-

NONLINEAR PHYSICS AND WERNER HEISENBERG

by Norman J. Zabusky

All physical processes are nonlinear if observed over a sufficiently long time or generated with a sufficiently intense source. For example, Einstein has remarked that Maxwell's equations in empty space represent formulations which are linear in form only because they are based on experience with very weak electromagnetic fields. "In general linear laws fulfill the superposition principle for their solutions but contain no assertions concerning the interaction of elementary bodies. The true laws cannot be linear, nor can they be derived from such."

It is too bad that this is a rather general feature of physical laws, because mathematical analysis of linear laws is relatively simple. Linearity implies solutions may be superposed; this principle permits reduction of a complex problem into simpler elements, a technique of enormous analytical power. On the other hand the elucidation of nonlinear phenomena is difficult. One often finds that methods of solution are highly restricted or "rigid" in that they are usually rendered useless if the problem posed is modified only slightly. The study of nonlinear problems has progressed slowly with few general results obtained compared to the progress of linear mathematics and physics.

During the past 50 years the major step forward in physics has been the development and application of quantum theory, a fundamentally linear theory. The name Werner Heisenberg is intimately connected with its basic concepts and further developments. It is probably not so well known that he also has had a recurrent interest in nonlinear physics problems.

His first published paper dealt with vortex motion, and two years later in 1924 he published a paper on the stability and turbulence of a flowing fluid and another on solutions for a nonlinear differential equation describing a viscous fluid. In 1936 he published a paper with H. Euler in which, using quantum theoretical methods, he derived the Lagrangian density for the electromagnetic field to the next, nonlinear, order. In 1948 three papers appeared on the statistical theory of turbulence and in 1953 a paper on "Meson Production as a Shockwave Problem." At that time his belief in the fundamental nature of nonlinear processes merged with his interest in applying quantum theory to elementary particles, and in 1953 he published a paper on quantization of nonlinear equations. This work has been followed by many papers by him and his associates on the quantum theory of nonlinear wave equations and nonlinear spinor theory.

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Condensed from an introduction to the International School of Nonlinear Mathematics and Physics.

cult to know how far such a theory can work. It is true that the motion of the planets can be followed rather easily by perturbational methods for a finite amount of time, say for a few thousand years. But when time increases, perturbations may no longer be small.

Resonance effects have been observed owing to commensurabilities between the orbital frequencies of different planets, and in the course of time there may be a big effect, so that in certain cases it finally may be impossible to say whether the orbit will be periodical or whether the planet will move out of the system. Such questions can be extremely difficult to answer.

I might mention a most paradoxical result of this mathematical analysis—the theorem of Heinrich Bruns. He proved that even in an infinitesimaly close neighborhood of a point where the perturbation theory converges, there must always be other points where the perturbation theory diverges. So one can say that the points where the perturbation theory converges and those where it diverges form a dense manifold. This result suggests that after a very long time one can never know where the orbit finally will go. It is only for a finite time that we can really determine what will happen.

This is not only a mathematical result interesting for the astronomers: it may have very important practical applications. I might just mention a problem that troubled us about 15 years ago when the CERN proton-synchrotron was constructed. For the big CERN accelerator it was very important to know whether a proton running around the circle of the accelerator would be stable in the assumed orbit or not. One could select conditions such that there would be stable oscillations around the orbit. But when there is some small perturbing defect for the orbit in the tube there may be resonances. The frequency of rotation of the proton around the machine and the frequency of oscillations around the orbit can become commensurable, and then there could arise difficulties similar to those in the astronomical problems.

One would expect the amplitude of the oscillation to increase if there is resonance, but then you come into the region where the problem is not linear and perturbation theory does not work. On account of the nonlinearity the frequency will be changed, and therefore the resonance will be stopped. In this way the nonlinear terms should exert a stabilizing influence until the motion has passed the critical region of resonance.

This was the hope. When numerical calculations were carried out, the result was that the particle may actually run around its orbit 10 000 times; that is, the stabilizing effect seems to work very well. First the amplitude increases, then the frequency gets out of resonance, then the amplitude decreases again, and so on. But after 10 000 revolutions the particle can run out of its orbit just as well. So you can get this kind of surprise, a final instability after a long time of apparent stability, in nonlinear problems, and that, I think, is a very characteristic feature of nonlinear problems. Therefore one might say, to use a very simple term, that nonlinear problems have a certain kind of unpredictability. One doesn't know how the solutions will behave after a very long time; I think that this may be a very general feature of nonlinear problems.

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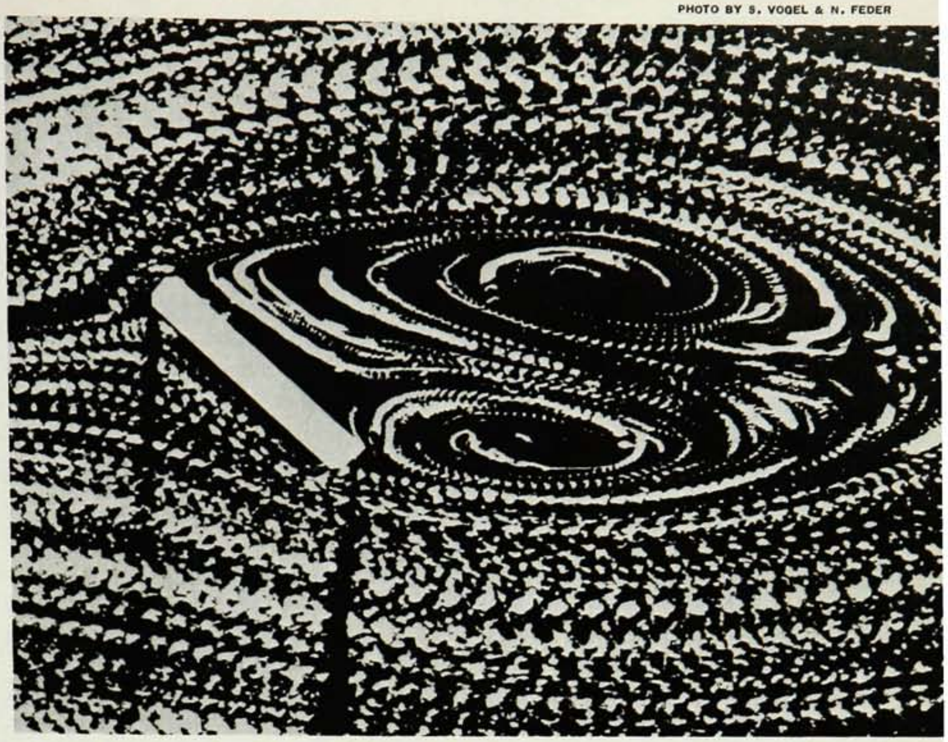


LAMINAR MOTION will be stable at small Reynolds numbers, but above certain values there are deviations for which the amplitude increases resulting in turbulent motion.

Statistical methods. What can one do if it is too difficult to study the single orbit, too difficult to get a kind of survey of all the possible solutions of such an equation? The way out is to investigate, not a single solution, but ensembles of solutions. One can argue that it may not be necessary to know every detail of the solution, that one can be satisfied with an incomplete knowledge of the system. In this case one cannot ask: What will be the state of the system after a certain time? Rather one has to ask: What will be the probable state of the system after some time? This has been the line of research in statistical mechanics. If you follow it, you may get into the difficult problem whether an average over the time variable is equivalent to an average over the ensemble. I only mention the problem of the ergodic hypothesis; I cannot go into any details here.

From this brief review of point mechanics we can conclude that we have four characteristic features that we will probably find in many nonlinear problems. First, we can always use the *symmetries* to simplify our problems. The second, *linearization* of the problem, is possible around special solutions that are simple and may be found just by looking at nature. The third is that solutions have a kind of *unpredictability*—you never know what the solution will finally do, and it may not be possible to extend the solution beyond a certain time. And finally, one possible way out is *statistical methods;* we can be interested, not in special solutions, but in ensembles of solutions or in systems that we do not know completely.

Continuous media

But let me now come to those problems of present interest. They are the problems with infinitely many degrees of freedom or problems concerning continuous media. Such problems, of course, exist in very great number—fluid dynamics, gas dynamics, elasticity, electromagnetism—electromagnetism is a nonlinear theory if the interactions with the bodies are taken into account—and finally gravitation.

And let me mention at once a complication which comes up again and again in this field of physics. Equations in which the field quantities depend on the continuous variables x, y, z and t are almost necessarily incorrect for the following reason: At very small distances one has to take atomic structure into account. Usually in hydrodynamics we need not do it. For example, in the Navier-Stokes equation we do not speak about molecular structure of a liquid, we just introduce a viscosity term. The same is true for elastic bodies, for problems of gases, and so on. Still, we have to remember that at very small distances something new will happen; something that is not described by the equation, and this has to be studied if the equation does not determine the course of events within the bigger dimensions. It is only in quantum field theory or in electromagnetism that the continuum may be a fundamental continuum. But even there our doubts arise, and there we come to the general question of whether quantum theory is a linear or a nonlinear theory.

Symmetry and linearization. Now we can try to see whether, in these many problems of fluid dynamics, elasticity, etc., one can use the same methods of simplification that I have mentioned in point mechanics. First of all, one will definitely always use the symmetry properties of the underlying system and the conservation laws. Then, one will use the method of linearization. How this linearization works I might just explain in one problem that I studied myself some time ago and in which I was strongly interested for many years—the stability of laminar hydrodynamical motion. At that time one studied mostly incompressible fluids, say, the flow of water, and so started from the Navier-Stokes equation, which introduces, besides inertia, also the viscosity of the liquid. The question was whether a laminar flow is stable or unstable.

One knew from the experiments that the laminar motions were only very special solutions of these equations. The general solutions were much more complicated; there we have to do with the big field of turbulent motions that one knows from daily experience. The laminar motions are known, one can say, from inspection. We see how a liquid can flow through a tube or between two parallel plates, and we can easily find the solutions of the Navier-Stokes equations corresponding to these simple laminar motions. They had been derived long ago and described in the textbooks. But such a solution is sometimes not stable. It is well known that when a liquid flows through a tube and exceeds a certain speed, the type of motion changes completely. Instead of the smooth flow through the tube, we have all kinds of vortex motion appearing: Eddies are formed and dispersed again, and one can ask why that happens at a certain speed. This was a typical problem that could be solved by the method of linearization.

One could ask for motions in a very small neighborhood around the laminar motion, and since this latter motion was stationary, the coefficients in the linear equations for the perturbation were constant in time. So one could look for periodical solutions and then solve an ordinary eigenvalue equation as in quantum theory. One could use those mathematical methods that later proved useful in quantum theory, namely the investigation of asymptotic solutions, or one could introduce the modern techniques of applied mathematics including the use of electronic computers. The result was that for small Reynolds numbers all deviations from laminar motion would die down, but above a certain Reynolds number there are deviations for which the amplitude increases, and then one has an entirely different type of motion, namely turbulent motion, that cannot be treated by perturbation theory.

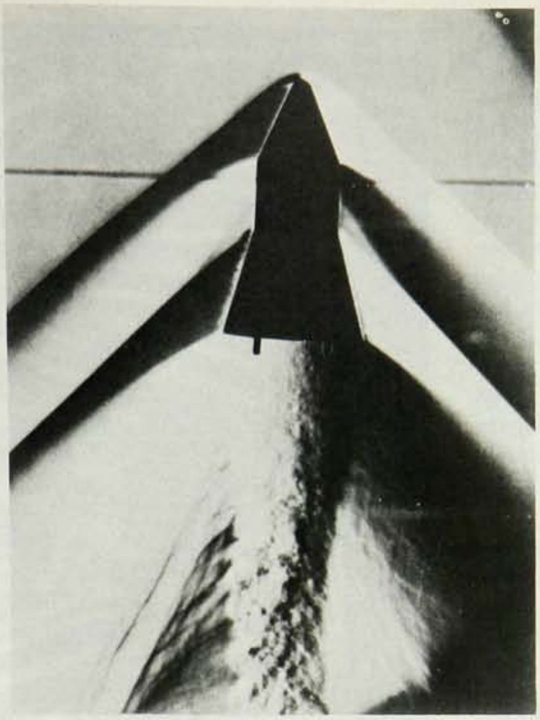
Unpredictability. But let me now come to really nonlinear problems in physics such as occur in gas dynamics and which are especially interesting. While in linear motion, for instance in linear wave motions like sound waves, the velocity is essentially independent of the amplitude and independent of the frequency of the waves, the situation is entirely different for a nonlinear wave motion. If, for instance, we have an initial discontinuity, then in the linear wave equation we would expect that this initial discontinuity, say across a surface, will be propagated with the velocity of sound like any other wave.

But in a nonlinear motion something different happens. This discontinuity may either disappear at once or may be propagated as a shock front—one or two shock fronts—not with the speed of sound but with supersonic speed. And not only that; it may be that out of a smooth motion in which there is no discontinuity at the beginning, a discontinuity will be formed later. I think that this fact corresponds in some way to the other fact mentioned before that nonlinear equations always have a tendency to diverge, that is, to reach points in time beyond which one cannot continue the solution.

One can say in a more mathematical language that nonlinear equations often do not admit solutions that can be continuously extended to any region where the differential equation remains regular. So the regularity of the equation does not guarantee the regularity of the solution, and you may after a finite time get to singularities in the solutions. If such a singularity occurs, the equation itself cannot determine what happens afterwards. At least the equation alone is not sufficient since we come to a point where the behavior of the substance in very small dimensions becomes important.

In hydrodynamics or gas dynamics the fundamental equation treats the gas or the fluid as a continuum, and this cannot be correct for very small distances; here we have to take the molecular structure into account. When our solutions become discontinuous, the molecular structure must play a role in the region of discontinuity.

Fortunately, in some cases in gas dynamics it is sufficient to know that in the region of discontinuity some irreversible processes must occur. The irreversibility will necessarily lead to an increase of entropy, and this result alone is sometimes sufficient to determine the course of the shock wave. This seems to be a rather lucky incident in gas dynamics. At least in principle I don't see any reason why it should be so. Fundamentally, one could imagine a need to go into the microscopic details and study the behavior of the atoms in order to say what happens. But, as I said before, sometimes one can be lucky and things go better than might be expected.



IN NONLINEAR MOTION a discontinuity may be propagated as a shock front, or one or two shock fronts, and not with the speed of sound but with supersonic speed.

But even when we come to problems in which the atomic structure plays no role—problems that are continuous by their very nature—even then we may be confronted with the fact that solutions cannot be continued over a certain time, that solutions can disappear into nothing or arise from nothing. I would like to mention one example that has worried me for some time in quantum field theory. Let us compare a linear relativistic wave equation, namely the well known Dirac equation, with another equation that comes from the Dirac equation if one replaces the mass term of the Dirac equation by a nonlinear term. These two equations are

$$\gamma_\nu \frac{\partial \psi}{\partial x_\nu} + \kappa\psi = 0 \qquad (1)$$

$$\gamma_\nu \frac{\partial \psi}{\partial x_\nu} + \psi(\bar{\psi}\psi) = 0 \qquad (2)$$

In a linear equation like equation 1 we know that when the solution has been zero everywhere for some time, it will never become different from zero.

It is of course true that solutions of the corresponding inhomogeneous equation can have a different behavior; we can construct the Green's function to this wave equation. Such a Green's function could then be either a retarded or an advanced Green's function and may be different from zero only in the future or only in the past. But for the homogeneous equation 1 the only solution that is zero for finite space-like distances and different from zero in the future or in the past is a solution given by the commutator in the quantum field theory; it is identical with the difference between the retarded and the advanced Green's function. In a certain sense this function can be called a relativistically invariant solution of the homogeneous equation 1.

For equation 2, however, you have both types of solutions; that is, you can have solutions that are zero for spacelike distances and are different from zero for the past and the future, but you also can have solutions for the same equation that start, so to speak, from nothing. The wave function may be zero for the past and all of a sudden start to become nonzero at the singular point and develop like the retarded Green's function in spite of the fact that there is no inhomogeneous term involved here. This emphasizes very strongly the strange behavior of nonlinear equations. There may always be situations in which solutions of a nonlinear equation cannot be continued into the future or into the past. By the way, this problem plays an important role in the discussion of the later question whether quantum theory is linear or nonlinear.

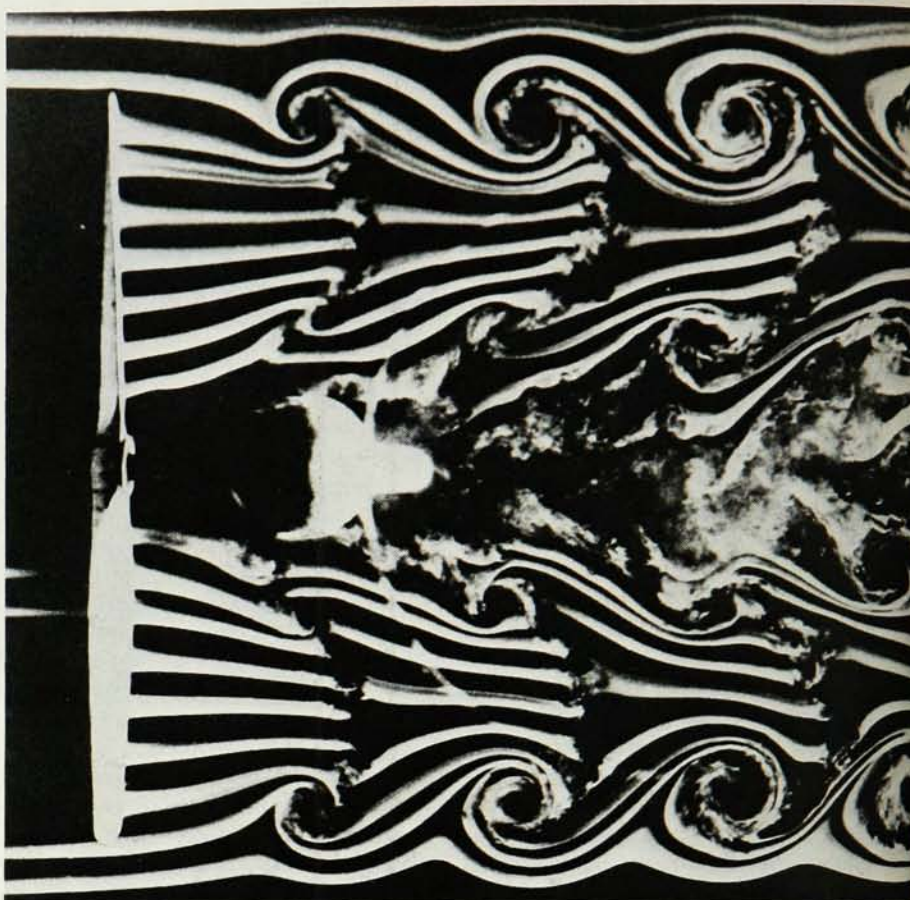
Statistical methods. Finally, we can treat nonlinear problems in hydrodynamics and gas dynamics with statistical methods as we do in point mechanics. As I said before, it usually is not possible to follow the complete set of solutions of a nonlinear equation. Therefore, it may be convenient to study ensembles of solutions, that is, the statistical behavior of fluids. The general principle of such a statistical approach is that one is satisfied with incomplete knowledge and incomplete description of the system; starting from this point one studies the probable development. For instance in the theory of turbulence, one can start with only the knowledge of the general distribution of the eddies. One knows only the spectrum of the motion, the intensity distribution of the different frequencies, without knowing the phase relations between these different amplitudes. In this case one can only ask what is the probable development of this liquid in the course of time.

Again at this point one gets into difficulties at very small instances, and in a certain sense we meet what in quantum theory we call the "ultraviolet catastrophe," because physically what happens is this: We have a turbulent motion, say in water, after we have produced by external motions some big eddies. These big eddies develop into smaller ones; so more and more smaller eddies are formed that get the energy. Finally the energy is dissipated into the infinitely many degrees of freedom that belong to the extremely small eddies. This process would go on to infinity if one did not introduce viscosity.

The concept of viscosity is, in this case, a nice way of getting around the fact that we get into the molecular region. Actually the energy is finally dissipated into thermal motion of the molecules, and this dissipation of energy into the motions of molecules can formally be replaced by including viscosity. But there may also be problems that can not be answered by the simple Navier-Stokes equation.

Generally I believe that in the problems of continuous media you will notice these four characteristic features already mentioned in connection with point mechanics. You will frequently use two essential simplifications, *symmetries* of the problem and *linearization*, which in many cases give a first insight into the problem. Then you will find the singularities, which result in the *unpredictability* of nonlinear solutions and frequently prohibit continuation of solutions beyond a certain point in time at which the singularity occurs. And finally, you will use *statistical methods* to get information not about a single system but about ensembles of systems, and thereby you will be able to predict either the probability for certain events or the probable course of events.

PHOTO BY F.N.M. BROWN



TURBULENCE in the wake of a propeller made visible by streams of smoke injected into the backwash of the propeller. Instead of smooth flow we have all kinds of vortex motion appearing.

Quantum theory: linear or nonlinear?

Is quantum theory a linear or a nonlinear theory? There is no doubt that quantum mechanics in its conventional form is a linear theory. It is linear in the sense that although the operator equations are nonlinear, they can be fulfilled by solving Schrödinger equations, that is, by looking for certain transformation matrices, and these Schrödinger equations are definitely linear equations.

Linearity in quantum theory has a very deep, almost philosophical reason and is not just connected with some approximation. In quantum theory we do not deal with facts but with possibilities: The square of the wave function describes the probability, and the superposition of wave functions, the possibility of adding two solutions to get a new solution, is absolutely essential for the whole foundation of quantum theory. Therefore it definitely would be wrong to say that the linear character of quantum theory is approximate in the same sense as the linearity of Maxwell's equations is approximate. The linearity of the quantum-mechanical equations is essential for the understanding of quantum theory and for the interpretation of quantum theory as a statistical basis for calculating what happens to the atoms.

This fact raises interesting mathematical problems on which I shall just touch. Let us, for instance, calculate solutions of the three-body problem, say, the helium atom, by means of quantum mechanics. We know that we get the complete system of all solutions by solving the linear Schrödinger equation in the coördinate space of the three bodies in the helium atom. So one could say that in quantum mechanics we have really solved the three-body problem that never has been solved in classical mechanics completely; and that seems somewhat strange, because in the limit $h \rightarrow 0$ the quantum-mechanical solutions must, if prepared in a certain way, go over into the classical solutions. Therefore, it seems as if we could replace the solution of a nonlinear problem in classical theory by solving linear problems in quantum theory and then going to the limit $h \rightarrow 0$.

Let me discuss the connection between the two theories a little further. The limiting process would require that we start our solution with a wave packet in coördinate space. We would put the two electrons around the nucleus at some approximate positions, moving with an approximate velocity so that the uncertainty relations are fulfilled, and of course these wave packets can be made smaller and smaller when we let the quantity h tend to zero, but for any finite value of h we would have finite wave packets. It is quite clear that the wave packet after a long time will disperse. It will become bigger and less dense and finally it will be spread out over the whole system. So we should always compare with this wave packet not a *single* solution of the classical problem, but instead an *ensemble* of solutions, in particular an ensemble belonging to the same set of uncertain initial conditions.

It is just the nonlinear character of the classical equations that makes the deviations between two neighboring solutions become bigger and bigger when time increases. In the classical theory, when we start with an ensemble of solutions belonging to a wave-packet-like distribution of initial conditions, after some time this wave packet will also spread out over large parts of coördinate space. So one can see that if we first let h and the size of the wave packet tend to zero, and then go to larger times, we do get a complete representation of the classical theory from quantum theory. If, however, we reverse the limiting processes, if we first go to infinite times and then let h tend to zero, the situation is quite different because then the wave packet has become infinitely big in every case.

In spite of this relation between quantum theory and the nonlinear classical theory, quantum mechanics is definitely a linear theory.

But is this still true when we come to the theory of elementary particles, a theory that does not start from given elementary particles but which tries to understand and to derive them? In our institute we are interested in an attempt called the nonlinear spinor theory. But I wish to emphasize that this term "nonlinear" does not necessarily mean nonlinear in the sense I have been using the word, for the following reason: We start from an operator equation which looks more or less like equation 2, and therefore we can call it a nonlinear equation. But the operator equations in quantum theory always are nonlinear equations and the question is whether the solution of these operator equations corresponds to a series of linear equations or nonlinear equations.

Generally we have learned that in quantum mechanics we can replace the nonlinear operator equation by a differential equation (Schrödinger equation) or a system of equations that are linear. In the same sense one could presume also that a nonlinear equation such as equation 2 can be replaced by a Schrödinger equation that in this case would be not a linear differential or integral equation but actually a linear functional equation. Alternatively, one could replace it by a system of infinitely many differential equations with infinitely many unknowns, and all these differential equations would be linear. This would be true if such a theory could be quantized along the ordinary rules of quantum theory. In this case we would have between $\psi(x)$ and (x') commutation relations that are essentially given by a delta function in space at the time $t - t' = 0$.

But this is still a controversial problem. From our present knowledge we can be rather certain that the commutator of such an equation does not look like such a delta function. Such a conventional commutator has a strong singularity, and this strong singularity just allows one to reduce a nonlinear problem of the operators to the linear problem of the Schrödinger equation. In field theory, however, we can take it for granted that in the center of the commutator we have no delta function because if we had a delta function, an equation like equation 2 would have no meaning. Therefore the singularity of the two-point function or the commutator at the origin is the real problem in any nonlinear field-operator equation like equation 2, and so we have to ask whether this problem can be treated with the help of delta functions in a similar way as in quantum mechanics.

If one formulates the equations for the commutator or the two-point function itself, these equations are extremely complicated nonlinear integral equations. If it should be necessary to solve these equations, the problem definitely would be a nonlinear problem, and at the very basis of quantum theory we would again come to nonlinear mathematics. The trouble is we do not know whether we really have to solve these equations.

It may be that the following procedure will be sufficient: We just guess approximate solutions for such a commutator and then put the chosen solutions into approximation schemes of the Tamm-Dancoff type; then of course in every finite approximation the result will depend upon whether we have taken a good or poor approximation of the two-point function. There are, however, examples which tend to show that in extremely high approximations the errors that we make by not having chosen the correct 2-point function, become less and less important, and it may be that even if we start with an incorrect commutator, at the end we get the right results in infinitely high approximation. Such at least is the situation for simpler problems in quantum mechanics that have been studied by Harald Stumpf and others.

Therefore it may be that again the actual treatment of nonlinear equations can be replaced by the study of infinite processes concerning systems of linear differential equations with an arbitrary number of variables, and the solution of the nonlinear equation can be obtained by a limiting process from the solutions of linear equations. This situation resembles the other one I mentioned where by an infinite process one can approach the nonlinear three-body problem in classical mechanics from the linear three-body problem of quantum mechanics.

Nonlinear progress

The present conclusion is that we do not know whether ultimately the fundamental problem in the quantum theory of elementary particles will be a nonlinear problem or a linear problem.

Finally I wish to emphasize again that the progress of physics certainly will depend to a large extent on the progress of nonlinear mathematics, of methods of solving nonlinear equations. It may still be that every such problem is individual and requires individual methods. Yet, as I have said, there are definitely some common features and therefore one can learn by comparing different nonlinear problems. □