

New Foundations for Quantum Physics

Attempts to understand better the basic principles of quantum physics continue. Because of the epistemological problems involved, the process is often controversial. The present article attempts to derive quantum mechanics from simple nonquantal postulates.

by Alfred Landé

SIXTY-SEVEN YEARS AGO Max Planck initiated the quantum age, but its conceptual content still mystifies the student. And the efforts to explain the basic mathematical rules of the theory by developing them from certain fundamental principles (wave-particle duality and complementarity) beg the question since those rules and principles themselves are of a quantum character. For this reason several theorists have gone one step further in their analysis and have established a system of axioms that do *not* include any quantum rule explicitly. However the axioms—I refer to the 14 theses of Fritz Bopp in the Heisenberg festschrift and those of Günther Ludwig in his *Grundlagen der Quantenmechanik*—are of a highly abstract mathematical kind and will hardly satisfy the physicist's desire for an explanation of the bewildering quantum theory.

As much as I admire those mathematically unassailable axiomatic systems, I do not think that they reveal

much of what is really behind the strange rules of calculation and paradoxical quantum antinomies. After studying those axioms and being imbued with the usual ways of interpretation, the student will practice the formalism merely as "the tricks of the trade," the more so as he is told that he ought to "understand that there is nothing to be understood in quantum mechanics,"¹ and that "the real problem was to refine the *language* of physics, rather than vainly trying to reduce one aspect [waves] to another [particles]."²

In contrast to being satisfied with



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Interconnection of the ψ 's and P 's

established ψ -theorem	group	$\psi_{AA} = 1$	$\sum_{\beta} \psi_{\alpha\beta} \psi_{\beta\alpha} = 1$
desired P -law	group	$P_{AA} = 1$	$\sum_{\beta} P_{\alpha\beta} = \sum_{\beta} P_{\beta\alpha} = 1$

nonunderstanding, I think that a deduction of the theory from *nonquantal* postulates is desirable. I hold, however, that to be explanatory the non-quantal postulates ought to be of a simple, plausible, almost self-evident kind—without giving an exact definition of these terms. At the same time, combination of the basic postulates ought to yield the well known nonrelativistic quantum formalism. The following new approach does not claim to be without blemish or to be complete. Yet it has brought, not only to me, the experience: "Now I begin to understand quantum mechanics." At any rate, the aim is to demystify a theory that for too long has had the reputation of incomprehensibility, going as far as the assertion that it involves a revision of ordinary (Aristotelian) logic.

Probability matrices

The discussion to follow starts by abandoning classical deterministic theory in favor of a statistical or probabilistic schema of a very general kind, later to be applied to mechanics. Thus, before dealing with energy, momentum and other physical quantities, I speak only of various "observables" A, B, C and so forth. The quantity A for a given mechanical system, may be capable of various values $A_1, A_2, \dots$. Whether these values form a continuous band or are discrete and then are denoted as characteristic or *eigenvalues* is irrelevant so far. For the sake of mathematical simplicity we assume at first that the *multiplicities* $M = m$ and $M = n$ in the series

$$A_1, A_2, \dots, A_m, \text{ and } B_1, B_2, \dots, B_n, \text{ etc.} \quad (1)$$

are finite pending later generalization. On the other hand, there may be an infinite number of observables A, B, C , etc. of a most diverse nature.

Suppose now that an instrument built for determining A values, called

an "A meter," has ascertained the system to be in one of the states A_{α} . When the system is now subjected to a B -meter test, sometimes the value B_1 , another time B_2 , etc., will turn up without possibility of individual prediction. The statistical proposition contains the assumption, however, that there is order such that the various states $B_1, B_2, \dots, B_n$ occur with certain statistical frequencies or probabilities

$$P(A_{\alpha} \rightarrow B_1), P(A_{\alpha} \rightarrow B_2), \dots, P(A_{\alpha} \rightarrow B_n) \quad (2)$$

the sum of which is *unity*. One may compile a rectangular array, a *matrix*, of all probabilities P connecting various original A states with subsequent B states revealed by a B meter

$$\begin{pmatrix} P(A_1 \rightarrow B_1) & P(A_1 \rightarrow B_2) & \dots \\ P(A_2 \rightarrow B_1) & P(A_2 \rightarrow B_2) & \dots \\ \dots & \dots & \dots \end{pmatrix} = P_{AB} \quad (3)$$

with m rows and n columns. Each *row* sums to unity. All this is almost self-evident as soon as one accepts a statistically described schema of events ascertained by the application of "meters."

The symmetry postulate

Far from self-evident is the assumption: *there is a two-way symmetry of the individual probabilities*

$$P(A_{\alpha} \rightarrow B_{\beta}) = P(B_{\beta} \rightarrow A_{\alpha}) \quad (4)$$

This postulate looks *plausible* enough as being the statistical counterpart of the time reversibility of classical deterministic processes.

From equation 4 it follows that the columns of the matrix P_{AB} above are identical with the rows of the matrix P_{BA} which, as rows, sum to *unity*. Therefore not only the rows but also the columns of P_{AB} sum to unity. And since the sum of all P 's of the matrix is the sum m of all rows and also the sum n of all columns, m must

equal n , so that the matrix P_{AB} must be *quadratic*, with the observable A having the same multiplicity m of values as the observable B and any other observable C and D and so forth in the same mechanical system. Omitting the arrows from now on and writing $P_{\alpha\beta}$ for short

$$\begin{aligned} \sum_{\beta} P_{\alpha\beta} &= 1 & \text{for every } \alpha \\ \sum_{\alpha} P_{\alpha\beta} &= 1 & \text{for every } \beta \end{aligned} \quad (5)$$

Notice the special case describing *reproducibility of a test result* (without which no objective physics would be possible)

$$P_{\alpha\alpha} = \delta_{\alpha\alpha'} \text{ or } P_{AA} = 1 \quad (6)$$

rendering (P_{AA}) a *unit matrix* (ones in the diagonal and zeros outside the diagonal).

From now on we denote as observables only such quantities A and B , etc., as fit into the schema above. Thus, position xyz is not an observable, but position xyz at time t_A is an observable A , and xyz at time t_B is another observable B although one can hardly observe the position of a particle inside an atom at any time t . The term "observable" is a misnomer, but established terminology cannot easily be changed.

Probability interference

The law of probability interference was originally found as a special feature of quantum mechanics, as a supplement to the Schrödinger equation, resting on a wave *analogy*. I intend to show that P interference is a natural consequence of the general probabilistic schema developed so far when the following postulate is added: *The various P tables (stochastic matrices) form a group connected by a single general operation, symmetric with respect to all observables $A, B, C, \dots$*

The emphasis here is first on the word "general." One can always construct, using his mathematical ingenuity, a group of P matrices of a special kind, for example, a set in which every matrix repeats the same elements in every row and column apart from zeros or has other high symmetries. Such solutions are unfit a priori to represent the various probabilities connecting pairs of observables of the most diverse sort. Yet despite the di-

versity of the individual P tables, their mutual interdependence is postulated to be *symmetric* with respect to exchanges among $A, B, C, \dots$. These exchanges are presented by a single repeatedly applied generating operator $\vee$

$$P_{AB} \vee P_{BC} \rightarrow P_{AC} \quad (7)$$

where $=$ indicates single valued and $\rightarrow$ multivalued determination under the operation $\vee$.

The requirement of generality involves exclusion of special solutions resting on mathematical intuition. Straightforwardness rather than artificiality is hard to define exactly, but it is vital in the construction of a credible physical theory. It is not a quantal restriction.

In the search for an operator $\vee$ satisfying generality and symmetry, restriction to multivalued determination ($\rightarrow$) is too vague to start with (it certainly would call for special ingenuity!). Let us therefore consider first single valued determination ($=$), applying it, however, to quantities named ψ which might later either be identified with, or at least be related to, the quantities P . The desired operation

$$\psi_{AB} \vee \psi_{BC} = \psi_{AC} \quad (8)$$

eliminates the "intermediate" B , which thus can be replaced by any other quantity D or E etc. Equation 8 satisfies the first condition necessary for the ψ quantities to form a group, namely associativity, just because it contracts AB and BC to AC .

The second condition for the quantities to comprise a group is that they include an identity member e such that $e \vee \psi_{AB} = \psi_{AB} \vee e = \psi_{AB}$. If $e = 0$ then $\vee$ symbolizes addition $+$. If $e = 1$ then $\vee$ symbolizes multiplication $\times$. Since the matrices P include identity members $P_{AA} = P_{BB} = \dots = 1$, only the alternative

$$\psi_{AA} = \psi_{BB} = \dots = 1 \quad (9)$$

can serve as a model for matrices equal or related to the matrices P . The procedure $\vee$ must therefore be symbolic multiplication

$$\psi_{AB} \times \psi_{BC} = \psi_{AC} \quad (10)$$

Likewise, we see that

$$\begin{aligned} \psi_{AC} \times \psi_{CB} &= \psi_{AB} \\ \text{and } \psi_{CA} \times \psi_{AB} &= \psi_{CB} \end{aligned} \quad (10')$$

follow from equation 10. However, the only consistent symbolic multiplication of matrices is the usual one of rows times columns, with equation 10 reading explicitly

$$\sum_{\beta} \psi_{\alpha\beta} \psi_{\beta\gamma} = \psi_{\alpha\gamma} \quad (11)$$

(Again, with mathematical ingenuity one might find other kinds of *symbolic* multiplication that might work for special kinds of ψ matrices that satisfy the group and symmetry conditions.)

The third condition for elements to form a group requires that each member have an *inverse*. This condition is satisfied; the inverse of ψ_{AB} being ψ_{BA} since equation 10 together with equation 9 yields

$$\psi_{AB} \times \psi_{BA} = 1 \quad (12)$$

or written out

$$\sum_{\beta} \psi_{\alpha'\beta} \psi_{\beta\alpha''} = \psi_{\alpha'\alpha''} = \delta_{\alpha'\alpha''} \quad (12')$$

In conclusion: The product theorem, equation 10, with the special case of equation 12 is the only way (within the framework of a theory excluding special and artificial constructions) of connecting quadratic matrices ψ including $\psi_{AA} = 1$ by a *single-valued* interdependence procedure with group properties *symmetric* in $A, B, C, \dots$.

The quantities ψ cannot be identical with the P 's since the latter are all positive whereas there must also be negative, or possibly complex, ψ 's to yield zero in the sum of products of equation 12' when $\alpha' \neq \alpha''$. And since the ψ theorem is the only single-valued one satisfying the conditions above, *the interdependence between the P 's can not be single-valued*. However, their connection can be related to the single-valued one among the ψ 's as seen from the table. The table suggests one simple and general solution of the multivalued P -interdependence problem, namely putting

$$P_{\alpha\beta} = \psi_{\alpha\beta} \psi_{\beta\alpha} = P_{\beta\alpha} \quad (13)$$

Hence either $\psi_{\alpha\beta} = \psi_{\beta\alpha}$ or more

generally, when admitting that the ψ 's might be complex ($*$ indicating the complex conjugate)

$$\psi_{\alpha\beta} = \psi_{\beta\alpha}^* \quad (\text{Hermiticity}) \quad (14)$$

Again Equation 13 is not the only mathematically possible solution of the P -matrix interdependence problem. But it is the only one at which one arrives in a simple straightforward way without appealing to mathematical intuition. It is the "natural" solution under the (perhaps somewhat metaphysical) assumption that *the basic laws of physical nature are simple*, and that complications are always a sign that one has not yet arrived at a fundamental level.

The reason that "Nature avails itself" of unitary transformations with complex quantities ψ , rather than being satisfied with orthogonal transformations of real ψ 's, will be seen in the next section.*

Mean and transition values

Probabilities are needed for the calculation of mean values. For example, the mean value of B over its eigenvalues B_{β} when starting from the original state A_{α} is defined as

$$\langle B \rangle = \sum_{\beta} P_{\alpha\beta} B_{\beta} \quad (15)$$

* The September 1966 issue of *PHYSICS TODAY* contains a critical review of my book *New Foundations of Quantum Mechanics*, a review intended to criticize my *derivation* of quantum mechanics from nonquantal postulates, because the *interpretation* had been dealt with by other critics before. However, with one exception, the review again concentrates on points of interpretation, general views about randomness, causality, plausibility of analogies, thought experiments (filters) and imperfections of language. Since these are primarily matters of taste, I do not see much value in arguing about them, the less so as they do not affect the main line of derivation.

The only exception is the critique of the argument concerning uniqueness of unitary transformations as solving the P -interdependence problems. I must confess that I, too, have regarded this part of the book as the weakest link in the chain of derivation. Therefore I am glad to offer in the present article a decisive improvement by using group concepts. (This development was first presented at the International Colloquium on Logic, Physical Reality and History, presided over by Allan Breck and Wolfgang Yourgrau, in Denver, May 1966.)

On the other hand, the reviewer's objection to the book version by way of a counter example (of a very special kind with P matrices consisting mostly of zeros and repetition of the same elements in all rows and columns) was justified because I did not emphasize enough the requirement of generality and physical appropriateness. These requirements seem to be implicit in the very sophisticated mathematical axioms of F. Bopp in the *Heisenberg Festschrift* (Vieweg, 1961). However, whether time is regarded as an observable or not is irrelevant for the close relation and analogous derivation of the two rules $p = h/\lambda$ and $E = h/\tau$ for systems with space and time periods λ and τ .

However, the critical review is welcome because it encourages improvements and may induce a mathematician to formulate the restrictions of "simplicity" and "exclusion of special unphysical counterexamples" more precisely.

Because of equation 13 the same quantity can be written

$$B_{\alpha\alpha} = \sum_{\beta} \psi_{\alpha\beta} B_{\beta\beta} \psi_{\beta\alpha} \quad (15')$$

We rename it "the mean value of B starting from A_{α} and returning to A_{α} again:" or more concisely "the transition value of B from A_{α} to A_{α} ." There is nothing new in this expression save the way of writing and the verbal description. However, within the framework of orthogonal or unitary transformations there is but one way of generalizing equation 15', namely

$$B_{\alpha\gamma} = \sum_{\beta} \psi_{\alpha\beta} B_{\beta\beta} \psi_{\beta\gamma} \quad (16)$$

which defines the "transition value of B from A_{α} to C_{γ} ." According to ordinary ideas, one would replace the quantities ψ in equation 16 with quantities P . But only equation 16 has equations 15 and 15' as special cases conforming to the general formalism of unitary transformations. If one defines the "matrix element"

$$B_{\beta\beta'} = B_{\beta\beta} \delta_{\beta\beta'} \quad (17)$$

then one can write equation 16 with a double sum

$$B_{\alpha\gamma} = \sum_{\beta\beta'} \psi_{\alpha\beta} B_{\beta\beta'} \psi_{\beta'\gamma} \quad (17')$$

When the ψ 's are real, $B_{\alpha\gamma} = B_{\gamma\alpha}$. In the more general case of complex ψ 's, the B 's like the ψ 's are Hermitian, as seen from equation 16.

$$B_{\alpha\gamma} = B_{\gamma\alpha}^* \quad (17'')$$

Wave function and selection rules

So far the discussion has been only about observables $A, B, C, \dots$ in general. Turning now to *mechanics* we have to do with special quantities q -position, p -momentum, E -energy, and t -time, forming pairs of "conjugates." In classical mechanics conjugacy is defined from the Hamiltonian equations of motion. But this definition of p and q as conjugates suffers from having to assume some chosen function $H(p, q)$ as representing the energy E conjugate to the time t . In quantum mechanics conjugacy of p and q is defined in terms of a relation between p and q themselves in three equivalent ways: by the Born commutation rule $pq - qp = h/2i\pi$, by the Schrödinger operator rule, $p \rightarrow (2i\pi)\partial/\partial q$, and by the wave function

$\psi(p, q) = \exp(2i\pi pq/h)$. All three introduce the action constant h associated with $i = \sqrt{-1}$. The quantum formalism was finally established, 26 years after Planck's $E = h\nu$ by the inductive genius of Max Born, Werner Heisenberg, Paul A. M. Dirac and Erwin Schrödinger. Today it is accepted because it works.

I propose here that it is a necessary consequence of combining the previous nonquantal assumptions (the two-way symmetry of the P 's and the postulate that the P -matrices form a group) with the nonquantal postulate of *Galilean invariance* in the following probabilistic version: *The mean value of any function $F(q)$ calculated for transitions from a state p to p'*

$$F_{pp'} = \int \psi(p, q) F(q) \psi(q, p') dq \quad (18)$$

(replacing the summation of equation 16 by integration since q has continuous eigenvalues) shall depend on the difference $p-p'$ only. And similarly, the transition value $G_{qq'}$ of any function $G(p)$ shall depend on $q-q'$ only. The same is true for the pair of conjugate variables E and t . The proof that these nonquantal postulates lead to the quantum theory runs as follows:

If the transition values of matrix elements $F_{pp'}$ are to depend on $p-p'$ for any function $F(q)$, one can take for F the special case of a delta function $D(q) = \delta(q-q')$ where q' is any chosen, fixed q value. In this case equation 18 reduces to the simple product

$$D_{pp'} = \psi(p, q') \psi(q', p') \quad (19)$$

The right-hand side cannot depend on $p-p'$ for any real function $\psi(p, q) = \psi(q, p)$. Rather $\psi(p, q)$ must be complex so that the second factor in equation 19 satisfies $\psi(q'p') = \psi^*(p', q')$ and ψ must have the *complex exponential* form

$$\psi(p, q) = a(q) \exp[ip\alpha(q)] \quad (20)$$

(where $\alpha(q)$ is real) since this is the only function rendering the product in equation 19 dependent on $p-p'$. Similarly, dependence on $q-q'$ of the

transition value $G_{qq'}$ of *any* function $G(p)$ requires

$$\psi(p, q) = b(p) \exp[iq\beta(p)] \quad (20')$$

where $\beta(p)$ is real). Comparison of equation 20 with equation 20' now leads to $a(q) = b(p) = \text{const.}$, $\alpha(q) = cq$ and $\beta(p) = cp$ (where the constant c is real) finally yielding $\psi(p, q) = \text{const} \times \exp(ipqc)$ or writing $2\pi/h$ for c

$$\psi(p, q) = \text{const} \times \exp(2i\pi pq/h) \quad (21)$$

where h is an "action constant" of the dimension pq . In a corresponding fashion one obtains from requiring *Galilean invariance* for E and t

$$\psi(E, t) = \text{const} \times \exp(2i\pi Et/h) \quad (22)$$

The expressions in equations 21 and 22 are wavelike functions of wavelength $\lambda = h/p$ and of frequency $\nu = E/h$, representing "probability amplitude functions" rather than "real" waves. The ψ functions in equations 21 and 22 have as their implications the Born commutation rule, the operator rule for p and E , the Schrödinger equation, the selection rules, and the uncertainty relation, in other words the main body of nonrelativistic quantum mechanics.³ There is no room here to show that *on nonquantal grounds* ψ functions of observables pertaining to two or more identical particles can only be symmetric or antisymmetric.

Since the argument presented here has been purely deductive throughout, one could, with considerable hindsight, imagine a theorist in his study beginning to draw up a probabilistic schema, then adding the two-way symmetry postulate and assuming a general interdependence law connecting the P matrices, and introducing Galilean invariance for the mechanical quantities p and q , as well as E and t . In this way he could arrive at the formalism of quantum mechanics without appealing to any experiment, or to the principles of duality or complementarity. $\square$

References

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