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DYNAMICS of PLASMAS

By David Finkelstein

LET us begin this sampler of elementary plasma dynamics by discussing one, most characteristic, mode of plasma motion, paradoxically called the "electrostatic wave". Suppose that all space were filled uniformly by free particles all of one charge q, with a number per unit volume n. These particles are found at time zero with various coordinates $\mathbf{x}$. Let us follow a small motion for a short time after time zero, designating their position at time t by $\xi = \xi(t,\mathbf{x})$. In this extremely simple situation, Newton's laws and one of Maxwell's equations suffice to give a complete solution of the problem. Newton tells us that for each of the particles

$$mD^2\xi = q\mathbf{E} \qquad (1)$$

(here D designates d/dt); Maxwell, that the electric field $\mathbf{E}$ arises from the charge density associated with the particles themselves:

$$\nabla \cdot \mathbf{E} = q\delta n. \qquad (2)$$

We can ask what charge density $q\delta n$ is created if the particle at $\mathbf{x}$ moves to the point $\xi = \xi(\mathbf{x})$. Evidently the source of the electric field depends on the gradient of ξ. For example, a uniform displacement $\mathbf{x} \longrightarrow \xi = \mathbf{x} + \Delta\mathbf{x}$ of the entire body of negative charge produces no net uncancelled charge, and produces no electric field. For small oscillations, the departure of the particle density from its average value is

$$\delta n = -n\nabla \cdot \xi. \qquad (3)$$

Now these three equations form an easily solvable system, because the last two are merely algebraic equations exactly in the case of uniform density:

$$\mathbf{E} = -qn\xi. \qquad (4)$$

Thus, just in this case of uniform density, we discover from (1) that:

$$D^2\xi + \omega_p{}^2\xi = 0, \quad \omega_p{}^2 \equiv -q^2n/m. \qquad (5)$$

The particles oscillate about their equilibrium position with uniform harmonic motion. The characteristic frequency ω_p of this uniform harmonic motion is called the plasma frequency, and the wave is called electrostatic, because at each instant the electric field is curl-free (conservative, longitudinal) just as in electrostatics. The really peculiar thing about this mode is that for various types of initial displacements, as characterized, for example, by their wave numbers, the frequency of oscillation is independent of the wave number. This makes the plasma quite different from the classical elastic media: *the unbounded plasma sings with absolute pitch,* while the classical elastic continuum will carry any note and acquires natural frequencies only through the action of boundaries (hence the violin, the tympanum, the organ pipe). This implies something about the group velocity with which such disturbances would propagate, at least in the infinite plasma for which this is valid: namely, they don't truly propagate at all. Just because the plasma frequency is independent of the wavelength of the initial perturbation, any small disturbance imparted to such an infinite plasma has zero group velocity and just stays where it is, oscillating harmonically with its characteristic frequency. This is evidently a stable mode.

When there are several species uniformly distributed in such an infinite volume, each with its own charge, mass, and speed, internal kinetic energy now stored in the plasma can go into an instability which can be thought of as a growing of these electrostatic waves. Suppose that, again to treat the simplest situation in which this happens, we consider two species, two stationary distributions of charged particles moving through one another. Again each of them obeys a Newtonian equation like (1):

$$m_iD_i{}^2\xi_i = q_i\mathbf{E}(\xi_i). \qquad (6)$$

(Now, however, the differentiation that appears must be carefully distinguished from the ordinary partial time derivative of this displacement ξ_i. We want the acceleration of the particle in question, whose position is instantaneously ξ_i. This acceleration requires us to perform twice the operation of so-called convective or dynamical differentiation, in which we move with the particle as we compute this derivative. So we have to put a species subscript $i = 1, 2$ on the operation of differentiation as well: $D \to D_i$.) And again we close the system by saying that the displacements give rise to the electric field that is acting on the particles themselves, in the same way as before, except that now a sum is required:

$$\nabla \cdot \mathbf{E} = \sum q_i \delta n_i \equiv \delta \rho, \quad \delta n_i = -n_i \nabla \cdot \xi_i. \quad (7)$$

Again, because of the uniformity, these last equations are solved algebraically:

$$\mathbf{E}(\mathbf{x}) = -\sum q_i n_i \xi_i. \quad (8)$$

Now it is best to deal with a sinusoidal initial perturbation, and the behavior will depend on the wave number of the initial perturbation. Suppose the perturbations in the positions are of the form

$$\xi_i - (x + v_i t) = \cdots \exp[-i(\omega t - \mathbf{k}\cdot\mathbf{x})] \quad (9)$$

where $\xi_i = \mathbf{x} + \mathbf{v}_i t$ describes the uniform motion. We will apply the usual naïve criterion that if either $\mathbf{k}$ or ω turns out to be imaginary when the other one is real then we have some sort of instability. The exact analysis of whether the instability is an amplifying or convective one or a true instability depends not on anything that can be learned by looking at one k and its ω, but on the entire form of the relation between ω and k.

Now, D_i becomes, aside from $(-1)^{\frac{1}{2}}$, just multiplication by the frequency of this small perturbation as seen by the i-th species. The frequency of the perturbation as seen from a point of rest is ω, but the frequency as seen from a point moving with the i-th species exhibits the Doppler shift associated with the velocity of the i-th species. Formally it follows from (9) that

$$D_i \equiv \partial/\partial t|_{\xi_i=\text{const}} = \partial/\partial t + (\partial \mathbf{x}/\partial t)|_{\xi_i=\text{const}} \cdot \partial/\partial \mathbf{x} \sim -i(\omega - \mathbf{v}_i\cdot\mathbf{k}). \quad (10)$$

Then (6) implies that the displacements are given by

$$\xi_i = -(q_i/m_i)\mathbf{E}/(\omega - \mathbf{k}\cdot\mathbf{v})^2.$$

Combining these, we can express the density fluctuation (7) in terms of the field producing it:

$$\delta\rho = \sum \frac{\omega_i^2}{(\omega - \mathbf{v}_i\cdot\mathbf{k})^2} \nabla\cdot\mathbf{E}, \quad \omega_i^2 \equiv n_i q_i^2/m_i.$$

Between this dynamical relation and the Maxwell relation (7) we eliminate $\mathbf{E}$, ρ:

$$\sum \frac{\omega_i^2}{(\omega - \mathbf{v}_i\cdot\mathbf{k})^2} = 1. \quad (11)$$

Each of the separate plasma clouds by itself would be capable of oscillating at the plasma frequency ω_i, we have said. This follows again from (11) when there is only one term in the sum. As seen by a particle in another cloud, this plasma frequency is Doppler shifted. The amount of the Doppler shift depends on the wavelength of the perturbation that each of the clouds is undergoing. Because the plasma frequency is independent of the wavelength, by taking a large range of wavelengths one can have a large range of Doppler-shifted frequency values: in fact one can assign to the Doppler-shifted frequency any value by taking an appropriate wave number. As a result, for two such charged clouds moving through one another, there exists a value of the wave number for which an electrostatic wave in species 1 coincides in Doppler-shifted frequency with one in species 2. (In the case of other oscillations, like sound, where the frequency is proportional to wavelength, this cannot always be asserted.) Such a coincidence of two separate frequencies typically results in a resonant transfer of energy between the two modes that are involved, and the dispersion relation that we have written down is just a typical dispersion relation for such a resonance. One verifies this as follows. For some given k, the left-hand side of the dispersion relation (11) looks like the curve of Fig. 1 in its dependence on ω, and the four intersections with the horizontal line are four real stable modes; two are derived from the plasma oscillations in the "electrons", two from the plasma oscillations in the "ions". But when ω_1 and ω_2 are sufficiently close, the picture changes to Fig. 2, there

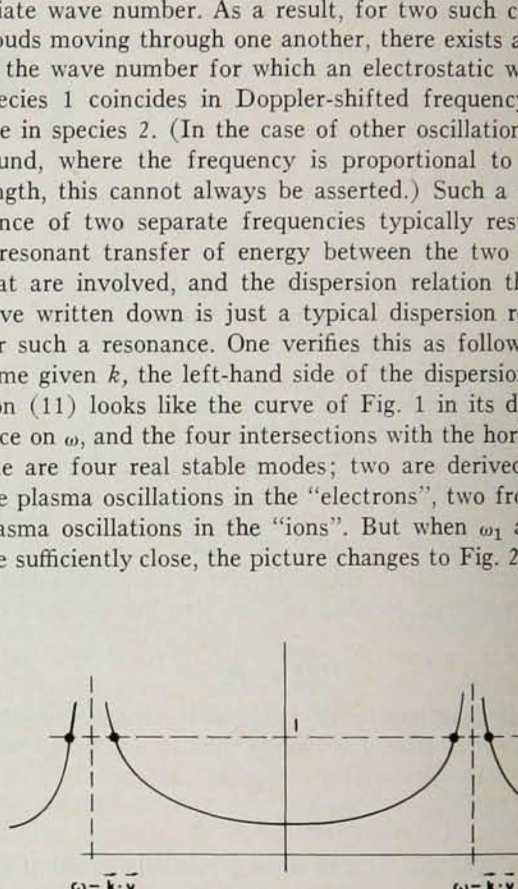

Fig. 1. Nonresonant electrostatic wave dispersion relation. Four real (stable) roots.

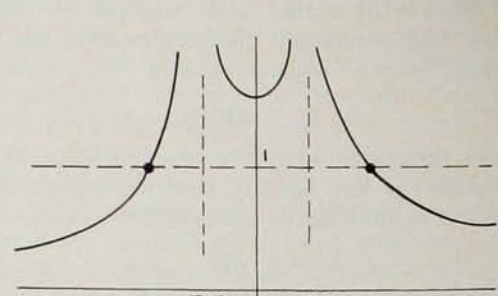

Fig. 2. Resonant electrostatic wave dispersion relation. Two real roots, hence two complex roots.

are only two real roots and therefore two imaginary, and the resonance described is taking place. The general result is that an internal anisotropy of velocity results in growing electrostatic waves of this kind. The anisotropy, however, must be sufficiently great: it is necessary for this instability that the distribution in some velocity component of the particles have two distinct maxima.

A closer inspection shows that one can avoid the growth of two-stream instabilities of this kind by taking advantage of the way in which plasma density appears in the theory. The fact that the ordinary plasma oscillation frequency is independent of wavelength results from the very simple interaction between two charge sheets in space, for a small sinusoidal perturbation of charge produces essentially a configuration of parallel sheets. The electric field produced by such a sheet is constant, and penetrates all space uniformly. If, however, the unperturbed plasma is a cylinder, then a sinusoidal perturbation in the density now produces, not infinite sheets of charge, but discs of charge. When these discs are far apart, the interaction between them is no longer independent of the distance between them, but rather behaves more and more like Coulomb's law, as the distance between them becomes large compared to the diameter of the cylinder. Accordingly, there is a reduction of the restoring force, for very-long-wave oscillations of the cylindrical plasma, and a reduction therefore of the plasma frequency for these very-long-wave modes. In fact, for very-long-wave perturbations, there is approximate proportionality between frequency and wavelength, and a true velocity of propagation appears. As a result, it may be impossible to find wave numbers for which this resonance arises, and in fact there are plasma columns of reasonable sizes that may not form two-stream instabilities.

Magnetic-Surface Waves

These growing electrostatic waves are a typical *bulk* instability of plasma. *Surface* instabilities of plasma were first treated, in a very pleasing and intuitive way, as follows: if one imagines an interface completely separating a plasma at a gas pressure p and a magnetic field B, then for mechanical equilibrium of this configuration, the forces exerted by the magnetic field on the skin currents flowing in the plasma, which cancel the magnetic fields within the plasma, must exactly balance the plasma pressure. It is easy to express the total stress exerted on the plasma in terms of the external magnetic field, eliminating the reference to the skin currents, and one then says the so-called *magnetic pressure* must balance the plasma pressure.

If, however, the plasma surface is curved, then there are regions away from the plasma where the magnetic field is either stronger or weaker, depending on the curvature of the surface. If the surface curves away from the plasma, the magnetic field is larger away from it; if the surface curves toward the plasma, the magnetic field is smaller away from it. The plasmas, being diamagnetic, are able to "stick out their tongues" at the experimenter in this case, protruding extensions into regions of weaker magnetic field by growing surface instabilities. This is the sort of argument that first led fusion physicists to consider magnetic fields having a configuration which curves away from the plasma over as large a region of the surface as is possible.

"String" Instabilities

Since we have illustrated some volume and surface modes of plasma dynamics, let us make some qualitative remarks about further modes that long, thin plasma columns exhibit. The thing is most easily understood in the case of very-long-wave instabilities. We recall that waves in a string propagate with a velocity that depends on the tension T in the string and the mass per unit length M of the string:

$$v = (T/M)^{\frac{1}{2}}. \qquad (12)$$

This is a consequence of the basic laws of conservation of energy and momentum for the string. The fact is that the string is capable of being deformed in a sort of geometrical way; we can deform it at right angles to itself and obtain a new configuration which is in fact a geometric deformation of the original one. At least approximately, all the physical quantities attached to the string, like the mass or tension, are just carried along under this transformation. In such a case, the ordinary law of conservation of energy and momentum for the perturbed column leads in general to an equation in which the original stress tensor T_{ij} of the column determines the "characteristics" of the partial differential equation according to which transverse displacement ξ develops:

$$T^{ij}\partial_i\partial_j\xi = 0.$$

Here it is strictly a two-dimensional problem: $i = 0, 3$. There is only time $x^0 = t$ and one spatial coordinate $x^3 = z$ along the plasma column, and T_{ij} is actually an integral over the cross section of the column. It is always possible (by choice of reference system) to write this as

$$M\partial^2\xi/\partial t^2 - T\partial^2\xi/\partial z^2 = 0$$

from which follows (12), where $T[= -T^{33}]$ is the tension and $M[= T^{00}]$ the linear mass density of the column. In these long-wavelength cases, where the internal stress configuration is only slightly affected by the displacement aside from being carried by the column, it follows that if the system is entirely in a state of tension—if the total force exerted by one part of the column on a neighboring part is directed toward itself—then harmonic spatial dependence implies harmonic time dependence; one has stability. If the column is in a state of compression, however, then these very long-wave oscillations grow in time; the system is unstable. This is, perhaps, an excessively complicated way of saying that if we pull on a thin column it is

stable, and if we push on it, it collapses. (In this, one is not taking into account any internal transverse structure of the system under consideration. This is appropriate, then, only for a very-long-wavelength disturbance, where the radius of the system is negligible compared to the wavelength of the disturbance.) For example, consider a "pinch": a magnetic field passing around a plasma column, caused by a current passing along the column. Observe that this magnetic field contributes a net compression rather than a tension in the plasma column, for the parallel lines of force repel each other (in the terms of Faraday). Moreover, the gas pressure also contributes a net compression along the column, so necessarily the entire stress in the plasma column is one of compression. Therefore, these columns, if sufficiently long, should undergo growing exponential deformations. This is what first led people to suggest running internal magnetic fields along the length of the pinch, because the contribution of this magnetic field to the internal stresses of the column is in the nature of a tension. As Faraday would have understood it, the magnetic-field lines tend to shorten as well as to repel each other.

This simple picture of the plasma column as a string can also help us to understand the effect of damping forces on instabilities. We might expect that if a system is theoretically unstable and we turn on damping forces acting to oppose any motion of the plasma through an external medium, then there is a possibility of stability. Such drag forces arise whenever there is a background gas in which the magnetic field carried by the moving plasma column can induce eddy currents.

Let us look at this just as a system of a string in an external viscous medium, neglecting the internal structure of the plasma except for the stress and mass per unit length:

$$(M\partial^2\xi/\partial t^2) - (T\partial^2\xi/\partial z^2) = - K\partial\xi/\partial t,$$

where K is the coefficient of the viscous drag; or

$$M\omega^2 + iK\omega - Tk^2 = 0.$$

Thus we rederive the long-wavelength limit of the already known analysis of this instability:

$$\omega = \frac{- iK \pm i(K^2 - 4MTk^2)^{\frac{1}{2}}}{2M} \rightarrow \begin{cases} +2iM|T|k^2/K, \\ -iK/M \end{cases}$$

for $k \rightarrow 0$, $T < 0$.

(The coefficient K is given approximately by

$$K = \alpha\mu_0{}^2I^2\sigma,$$

where α is a dimensionless geometry factor of order unity, I is the plasma current, and σ is the background conductivity. For large ω the full complex conductivity σ should be used; this can significantly improve stability.[1]) For $T < 0$, the lower choice of sign gives the kind of dying-out expected of viscously damped systems; the upper sign gives a potentially disastrous exponential growth in time.

[1] Remark of W. H. Bennett and G. Massel, unpublished.

Phase-Telescope Experiment

Let me say a word about some attempts we are making in our laboratory to study the evolution of the dynamic behavior of plasmas experimentally. One of the usual procedures for getting information about the interior of a plasma is to pass microwave radiation through it. At frequencies above the plasma frequency, the waves suffer a phase shift which provides a very sensitive measure of the electron density within the plasma. At frequencies below the plasma frequency, the wave is strongly absorbed. For extremely dense plasmas, one quickly finds that the necessary frequencies are not easily generated.

The instrument that we are developing may shed light on such dense plasmas, which are not open to ordinary microwave probing; for it employs light—micron waves instead of millimeter waves. The name "phase telescope" has been suggested for it, because it uses the same principle as the phase microscope.

The plasmas of interesting densities, even though opaque to microwaves, are, however, extremely evanescent objects from the point of view of ordinary light. A light wave passing through a meter of such a plasma is deviated by an extremely small angle and absorbed to an even smaller extent; the indices of refraction of these plasmas are very close to unity at optical frequencies, because the index of refraction approaches 1 as the ratio $\omega/\omega_p \rightarrow \infty$. Therefore, in order to observe these very tenuous objects, these so-called phase objects, special techniques are necessary.

One of these, which has been used very successfully in gas dynamics and is responsible for much of our experimental understanding of shocks in ordinary gases, is the so-called "schlieren" technique. There a point source is used to illuminate the gas, shock wave, or whatever object is being observed. All the light that passes through the object is brought to focus on a stop—often a razor blade—that obstructs all light which passes through the plasma without being deviated. Some of the light that is refracted by the object passes the obstacle and then enters another lens which casts an image of the illuminated object on a final screen. This schlieren technique, however, cannot be extended to very low plasma densities; evidently for very small indices of refraction, the image on the stop of the point source moves by a very small amount and there is a very small change in the image on the final screen. In order to extend such techniques to very low angles of refraction, it is therefore necessary to use sources of extremely small aperture. As one reduces the aperture of the source, one encounters diffraction-limiting: the image of the point source on the obstacle cannot be made smaller than a certain amount, depending on the optics of the system; and brightness-limiting: as one reduces the size of the source, the amount of light available for the photograph decreases. Since one is photographing an object which is luminous to begin with, and what is wished is information which is independent of the traditionally unreliable self-luminosity of the plasma, this is a serious restriction.

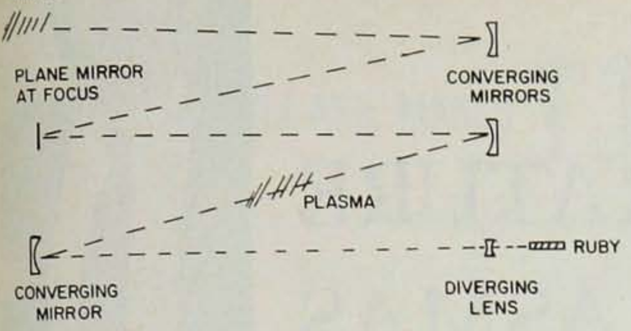

Fig. 3. Phase telescope. Parallel light from laser is passed through plasma and brought to a focus on plane mirror, which has a reflecting coating $\lambda/8$ thick. A hole burnt in this coating by the laser on a previous shot acts as a quarter-wave plate. Interference between light whose phase is shifted by this hole and light which is reflected from the coating provides intensity variation in the image.

Recently, however, extremely bright point sources have become available; the so-called optical maser, or laser, can pulse large amounts of light energy into a small solid angle. In the experiment that we are carrying out now, a high-power laser (operating at a megawatt power level and discharging in 1/10 of a microsecond) is indeed used to illuminate a dense plasma.[2] However, this description of the schlieren method makes it clear that it observes angles of refraction, and it observes, therefore, *gradients* in the index of refraction of the plasma. This is actually one of a family of several optical methods of observing tenuous objects; another, the phase method, responds directly to the index of refraction rather than to its gradient, and yet another, the shadowgraph method, responds to the second gradient, the second spatial derivative of the index of refraction. For us, the most interesting of these is the phase method shown in Figs. 3 and 4. The angles of refraction of visible light passing through plasmas of densities, say, of 10^{14} to 10^{16} particles/cm^3 and of laboratory size are extremely small, smaller than a milliradian, and could well be masked by the diffraction of the point source. However, there can be phase shifts of several waves in such a dense plasma, and it is exactly these phase shifts to which the phase-contrast method responds. The useful thing about this technique is that it should enable one to obtain simultaneously information about the entire cross section of the plasma; that is, to obtain a fairly detailed picture of its internal structure, instead of an average density measurement. In brief, one should be able to take a snapshot of the electron density.

This is an informal and preliminary description of an experiment now in process. In this way, one of many others reported at this meeting, we hope to learn more about the inner workings of plasma dynamics.

[2] The laser with which we began this work was designed and loaned by Dr. R. Daly of TRG, Inc., and his assistance in this work was extremely valuable. We have subsequently also used one of Trion Instruments, Inc. Mr. H. Presby is now engaged in this study.

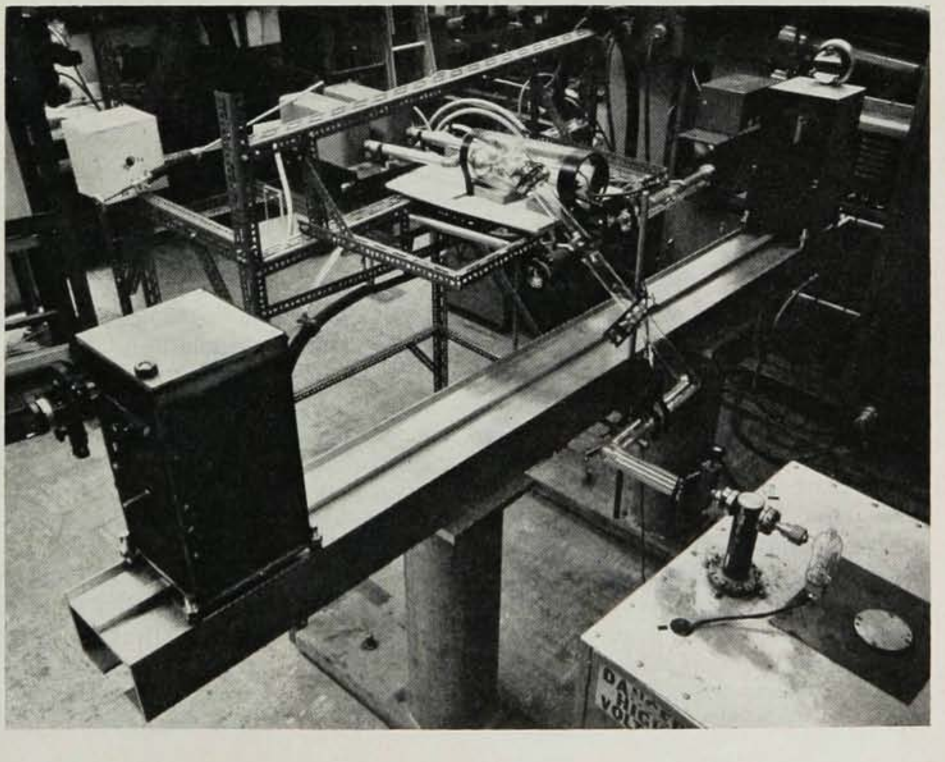

Fig. 4. Set-up of the phase telescope. Laser and half of the optics is in the far carriage; other half of optics is in the rear carriage, and the plasma system is between.