Hotel Management in ERGODIA

By John D. Trimmer

DURING an airport stopover in Europe recently, I had the pleasure of seeing again my two friends, the twin brothers Fritz U. Einwand and Hans W. Einwand. They still looked alike, to the same degree that used to confuse me when I saw them across the net of a tennis court in Ann Arbor—and to the same degree, no doubt, that prompted their parents to give them their humorous middle names.[1] I learned that they had become famous in the field of hotel management by contributing a sound mathematical development, based on studies made in their home country, Ergodia. The twins kindly loaned me a copy of a recent comprehensive report on their work.

Since one of the exciting prospects of our century is the possibility that the physical sciences may gradually build up that mathematically rigorous exactitude which characterizes the social sciences, I feel sure that physicists will find interest and challenge in a brief review of results achieved by the Einwand brothers. It is not too much to hope that similar developments in physical theory may be imminent.

Ergodian Hotels

Ergodia is a country much visited by tourists, and generally in a manner far from causal. Indeed, there is the old Ergodian legend that any given spot in the country will be visited at least once by any given tourist. As might be expected, then, the country is well supplied with inns and hotels. On the border between the vineyard country and the birch forests, for example, there is the venerable Birke-hof, which has more recently become well known also for its own wines, marketed under the label "Birkehof Gäret".

Although the Einwands' studies have some relevance to such provincial hostelries, the essence of their work is an intensive comparison of the three metropolitan hotels in the capital, Gibbsville. Here are located the Maxwell-Boltzmann House, famous for its coffee; the Bose-Einstein Plaza, renowned for gemütlichkeit; and the exclusive Fermi-Dirac Towers. These three establishments follow rather different room-assignment policies, and the Einwand brothers, using probability theory, have ingeniously analyzed the social and economic consequences of these differing house rules.

Some of the simpler results will be presented here. Let us begin with the distribution of guests by floors. This is especially important, because of the Ergodian law requiring all hotels to charge the same rent for all rooms on a given floor. Suppose the hotel has F floors and R rooms, with r_i rooms on the i-th floor. A distribution-by-floors of N guests is thus a set of F numbers, $n_1, n_2, \cdots n_i, \cdots, n_F$, subject to

$$\sum_{i=1}^{F} n_i = N. \qquad (1)$$

If the room rent[2] on the i-th floor is E_i, the total room expense for the N guests is E, given as

$$\sum_{i=1}^{F} n_i E_i = E. \qquad (2)$$

For the three hotels, the probability that a given guest will be assigned to a room on the i-th floor is a function of r_i and of the (momentary) value of n_i—that is, of the number of rooms on the floor and of the

[1] For those not familiar with the hotel trade, "Umkehr" and "Wiederkehr".

[2] Counted in ergs, the standard Ergodian monetary unit. Ergodian aristocrats prefer to deal in "jewels", one jewel being worth ten million ergs; and native tribes still use the foot-pound, obscurely related to the British pound.

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number of guests already on that floor. Given this probability, expressing the particular room-assignment policy, one can then write the probability of any given distribution. For the three hotels we denote the probability of a distribution (by floors) as P_{BE}, P_{MB}, P_{FD}. In each case, one can find

$$P = c/C \tag{3}$$

where c and C are numbers of distinguishable ways of assigning guests (identifiable for MB, unidentifiable for BE and FD) to identifiable rooms, with c referring to distributions over floors, and C to distributions over the entire hotel. Thus the c's are functions of n_i and r_i, and the C's of N and R.

The exact expressions are:

$$c_{BE} = \prod_{i=1}^{F} \frac{(r_i + n_i - 1)!}{n_i!(r_i - 1)!} \tag{4}$$

$$c_{MB} = N! \prod_{i=1}^{F} \frac{r_i^{n_i}}{n_i!} \tag{5}$$

$$c_{FD} = \prod_{i=1}^{F} \frac{r_i!}{n_i!(r_i - n_i)!} \tag{6}$$

$$C_{BE} = \frac{(R + N - 1)!}{N!(R - 1)!} \tag{7}$$

$$C_{MB} = R^N \tag{8}$$

$$C_{FD} = \frac{R!}{N!(R - N)!} \tag{9}$$

The probabilities expressed by eqs. (3) to (9) are based on postulates of equal a priori probabilities of the pertinent distributions—"complexions" or assignments of identified guests for P_{MB}; and "distributions-over-rooms", or assignment of unidentified guests, for P_{FD} (no more than one per room) and for P_{BE} (any number to a room).

It is interesting to note that all three probabilities given by eqs. (3) to (9) can also be written as the product of two factors, the first of which is the permutation factor, $F_p = N!/\prod_i n_i!$. The remaining factor in each case can be recognized as the probability of the given distribution-by-floors of *identified* guests, according to the "house rule" of the particular hotel. These rules can be understood by visualizing the N guests to be identified by their places in a waiting line, as they approach the registration desk. In the Fermi-Dirac Towers, the probability that any one guest as he comes up for registration will be assigned to a particular floor is a fraction having as numerator the difference between number of rooms on that floor and number of guests already assigned to that floor, and as denominator the same difference applied to the entire hotel. At the opposite extreme, the Bose-Einstein Plaza, dominated by a policy of togetherness, assigns guests to a given floor with a probability having as numerator the *sum* of number of rooms on the floor and number of guests already assigned, and as denominator the same sum applied to the entire hotel. Between these extremes the probability used by the Maxwell-Boltzmann House is the simple room-fraction, r_i/R, independent of guest assignments already made. Application of these rules is seen to lead, in the respective cases, to P_{FD}/F_p, P_{BE}/F_p, and P_{MB}/F_p. This alternative way of viewing the probabilities given by eqs. (3) to (9) makes clear that in all three cases the probability is for a distribution-by-floors of unidentified, or so-called "indistinguishable", guests.

The Fixed-Expense Problem

The most basic further result established by the Einwand twins is the solution of the problem which for brevity we denote as "fixed-expense". This is the problem, quite important in these days of limited group-cost tours, of finding the most probable distribution of N guests when the total group room expense is fixed. Maximizing of the appropriate probability (3), subject to constraints (1) and (2) added in with Lagrange multipliers α and $-\beta$, leads[3] to the solutions, which may be expressed for all three cases by means of the factor $\sigma = 0$ (for MB), $+1$ (for BE), -1 (for FD):

$$n_i^* = r_i(e^{-\mu+\beta E_i} - \sigma)^{-1} \tag{10}$$

where the asterisk on the n_i denotes the most prob-

[3] This simple method involves maximizing ln P, using the Stirling approximation—which, considering the large size of the Gibbsville hotels, would seem to be acceptable. Along with this approximation, the BE factor, $n_i + r_i - 1$, is approximated as $n_i + r_i$; and $N + R - 1$ as $N + R$.

able values, and where the quantity μ is related to the multiplier α thus:

$$\mu = \alpha - \ln\left(\frac{R + \sigma N}{N}\right). \quad (11)$$

In the Maxwell-Boltzmann case, μ can be eliminated by application of eq. (1), giving instead of (10):

$$(n_i^*)_{\mathrm{MB}} = N \frac{r_i e^{-\beta E_i}}{\sum_i r_i e^{-\beta E_i}}. \quad (12)$$

In the other two cases, eq. (1) also relates α to N, but in an implicit fashion not readily adapted to elimination of α. The other Lagrange multiplier, β, will be shown, in the concluding section below, to be related to temperature.

The Off-Season Theorem

Another interesting result is that when for all floors $n_i \ll r_i$, as may be expected during the off-season when the tourist arrivals are minimum, the "extreme" probabilities, P_{BE} and P_{FD}, converge to the middle-of-the-road P_{MB}. Algebraic methods suffice for showing this, and a helpful start is provided by Euler's result quoted on page 94 of the second edition of the hotel manager's mathematical handbook, edited by the well-known Gibbsville scholars, Margenau and Murphy.

The Disorder Theorem

A result which has generated much discussion and some controversy is the famous theorem on the increase of disorder.[4] The symbol for disorder is S, which many authorities believe to be derived from the initial letter of the colloquial German noun Schussel, meaning "a careless, slovenly person". Disorder is defined by the equation:

$$S = K \ln c \quad (13)$$

with the appropriate value of c to be chosen from eq. (4), (5), or (6). The significance of the constant K will become evident below in the discussion of temperature.

Since the disorder theorem makes a statement about the time derivative of S, we must preface the proof with a definition of the conditions under which S is considered to be changing. Although proof may be given in a broader context, we choose for brevity of presentation here the limitation to constant N and E, as expressed in eqs. (1) and (2). Then, with the same approximations used in deriving eq. (10), one gets

$$\frac{dS}{dt} = -K \sum_i \frac{dn_i}{dt} \ln\left(\frac{n_i}{r_i + \sigma n_i}\right). \quad (14)$$

The quantities dn_i/dt are restricted by the requirements (1) and (2), with the net effect that change from one floor to another can never be permitted for a single guest, but requires the "collusion" of two guests. If these are from the i-th and j-th floors, they may be reassigned to (or replaced by new guests assigned to) the k-th and l-th floors, provided

$$E_i + E_j = E_l + E_k. \quad (15)$$

Then the total change of numbers on a given floor is the sum of terms, which may be denoted $\dot{n}_{ij\to lk}$, signifying the number per unit time leaving floors i and j and appearing on floors l and k. The detailed expression for such terms is

$$\dot{n}_{ij\to kl} = A_{ij\to kl} n_i n_j (r_k + \sigma n_k)(r_l + \sigma n_l) \quad (16)$$

where the factors in parentheses represent the effect of the room-assignment policies of the several hotels (cf. the "alternative" discussion following eqs. (3) to (9) above), and the first three factors are the probability of transition from the i-th and j-th floor combination. Quantities represented in (16) by $A_{ij\to kl}$ are known as "collusion frequencies," and are the subject of an important assumption. Since, aside from the other four factors in (16), the collusion frequency constitutes the probability of the (double) transition in question, it is important to know whether these frequencies are symmetrical in the sense represented by the equality

$$A_{ij\to kl} = A_{kl\to ij}. \quad (17)$$

Whether this equality can be proved on the basis of other commonly made assumptions is perhaps a moot question. The Einwand brothers prefer to accept eq. (17) as a basic assumption.

The proof[5] of the disorder theorem is now straightforward. It is clear that dn_i/dt is the excess of $\dot{n}_{kl\to ij}$ over $\dot{n}_{ij\to kl}$, appropriately summed over all j, and over all *pairs* of index values, kl, compatible with eq. (15) and such that $k \geq l$. That is,

$$\frac{dn_i}{dt} = \sum_{j,kl} (\dot{n}_{kl\to ij} - \dot{n}_{ij\to kl}). \quad (18)$$

The remaining two steps in the proof involve

[4] The Ergodian Government has recently appealed to the Hotel Managers Association to minimize the use of technical terms conveying an unfortunate image to the public, particularly to the foreign tourist trade. Besides "disorder", the Government questions the use of "degeneracy", which has somehow come to denote the number of rooms per floor. There has also been muttering about racist implications in the term "complexion". As yet, however, no substitutes for these usages have become well established.

[5] Evidently the influence of hotel management theory on physics has already begun, as the discussion on pp. 368–370 of a widely-known physics treatise by D. ter Haar bears a remarkable resemblance to the Einwand proof reproduced here.

manipulation of summation indices. The first consists in replacing the sum over separate i and j indices, resulting from substitution of (18) into (14), with one-half the sum over *all* the ij-pairs. In general, if $B_{ij} = B_{ji}$, then

$$\sum_{i=1}^{F} \sum_{j=1}^{F} B_{ij} L_i = \tfrac{1}{2} \sum_{ij} B_{ij}(L_i + L_j).$$

Applying this to the combination of (18) and (16) in (14), and making any needed use of (17), one may write:

$$\frac{dS}{dt} = -\frac{K}{2} \sum_{ij,kl} A_{ij\to kl} [n_k n_l (r_i + \sigma n_i)(r_j + \sigma n_j) - n_i n_j (r_k + \sigma n_k)(r_l + \sigma n_l)] \times \ln \frac{n_i n_j}{(r_i + \sigma n_i)(r_j + \sigma n_j)}.$$

Now the second manipulation is to consider this expression with the ij and kl indices interchanged. It remains equally true, and so the value of dS/dt may be considered to be the average of the two forms (i.e., half of the sum). This average is

$$\frac{dS}{dt} = -\frac{K}{4} \sum_{ij,kl} A_{ij\to kl} [n_k n_l (r_i + \sigma n_i)(r_j + \sigma n_j) - n_i n_j (r_k + \sigma n_k)(r_l + \sigma n_l)] \times \ln \left[\frac{n_i n_j (r_k + \sigma n_k)(r_l + \sigma n_l)}{n_k n_l (r_i + \sigma n_i)(r_j + \sigma n_j)} \right].$$

But the product of the form $(x - y) \ln [y/x]$ cannot be positive and therefore the entire expression (since K and $A_{ij\to kl}$ are by definition positive) cannot be negative. Thus we have the result

$$\frac{dS}{dt} \geq 0. \tag{19}$$

Considering the basic significance to the hotel industry of the concept of disorder, it is understandable that the result (19) is most often referred to simply as *the* H-theorem.

Other Results

It would be possible to continue at some length these extracts from the Einwand treatise on hotel management. It would indeed be fitting to pursue the H-theorem further, especially in its implications as to equilibrium, fluctuations, and reversibility, since it was in resolution of these thorny questions that the Einwand brothers played a decisive historical role. But to keep this review within suitable limits, we shall conclude with just a few remarks about the effects of temperature on the Ergodian hotel business, directed particularly to identification of the Lagrange multiplier β.

In warm weather the more expensive upper-level floors of the Gibbsville hotels are more comfortable, and, as a result of the preference for these floors, the average room expense tends to increase with temperature. With temperature measured on the Kelvin scale, this tendency is well approximated as a linear proportion thus

$$\bar{E} = \frac{E}{N} = KT. \tag{20}$$

A second temperature effect which has been well established by observation is that the standard deviation of room expense, $\sigma_E = [\langle (E_i - \bar{E})^2 \rangle_{av}]^{\frac{1}{2}}$, tends to be of the same order as $\bar{E}$ and so can be well approximated by

$$\sigma_E = KT. \tag{21}$$

Differentiation of $\bar{E}$ with respect to T, considering rate-structure (values of E_i) to be constant, and using eq. (12) to express $\bar{E} = \sum_i (n_i^*/N) E_i$, readily leads to

$$\frac{d\bar{E}}{dT} = -\frac{d\beta}{dT} (\langle E^2 \rangle_{av} - \bar{E}^2) = -\frac{d\beta}{dT} \sigma_E^2. \tag{22}$$

The use of the most probable values, n_i^*, is matter-of-fact in regard to $\bar{E}$, because of eq. (2). But in deriving eq. (22) one must also accept $\langle E^2 \rangle_{av} = \sum_i (n_i^*/N) E_i^2$, which is in general an approximation.

Combination of eqs. (20), (21), and (22) yields $(d\beta/dT) = -(1/KT^2)$ or

$$\beta = 1/KT. \tag{23}$$

Because it is the Maxwell-Boltzmann distribution, as given in eq. (12), that permits the simplest derivation of eq. (23), the constant K has become known as the Boltzmann constant. Its numerical value is usually estimated as 1.4×10^{-16} ergs per degree Kelvin.

The omission of any integration constant from eq. (23) is a point which I have not found explicitly mentioned in the Einwand treatise. Presumably this is a detail that could be clarified in the context of the broader theory which includes variation of the rate structure. This more complete theory suffices, as the Einwands show, to explain the appearance of the same constant in eq. (20) and in the definition of disorder, eq. (13). These matters cannot be pursued in this brief survey, but it is hoped that the samples quoted above may encourage the more adventuresome physicists to a fuller exploitation in their own discipline of the advantages of mathematical reasoning.