THE FLUCTUATION DISSIPATION THEOREM

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1. Black-Body Noise in Black Boxes

IN 1928, J. B. Johnson [1] showed experimentally that a resistor with no current flowing in it has a measurable electrical noise across its terminals. The signal of the magnitude of a few microvolts is called noise because when made audible through hi-fi audio amplifier and speaker, it has a high-pitched hissing quality. Noise from a current-free metallic resistor is called Johnson noise, or sometimes thermal noise, because Johnson identified the voltage fluctuations across the resistor with thermal agitation of the charge carriers. His classic experiments showed that the mean-square-voltage noise signal was directly proportional to the resistance and the absolute temperature in various types of solid as well as in liquid resistors.

The first theoretical calculation of thermal-noise voltage in a resistor was reported by H. Nyquist [2] in 1928, the same year that Johnson first reported his experiments. Nyquist's theoretical approach was truly remarkable, and resulted in a fundamental equation of physics. He considered a resistor R connected to another similar and equal resistor through a lossless transmission line. The ideal transmission line thus permits one to use thermodynamic arguments and apply the equipartition principle. The resistors may be thought of as black boxes about which it is known only that they have an ohmic resistance, R. By calculating the power emitted and absorbed by each resistor in thermal equilibrium with the transmission line at a common temperature, Nyquist succeeded in showing that the spectral density $S(\nu)$ measured as a mean-square voltage is:

$$S(\nu)d\nu = 4kTRd\nu, \tag{1}$$

where k is Boltzmann's constant, T is the absolute temperature, and $d\nu$ is the bandwidth of frequencies measured. Likewise, the total power $P(\nu)$ available in the bandwidth $d\nu$ is: $P(\nu) = kTd\nu$. Equation (1), known as Nyquist's theorem, has been substantially verified by experiment. Moreover, in recent usage it has been generalized in its theoretical applications and called the fluctuation dissipation theorem.

The fluctuation thermal voltage arises only in the resistance and not in the shunt capacitance hidden in the resistor. Early experiments on capacitors in series with resistors where the resistor was kept at room temperature and the capacitor temperature varied showed that the capacitor did not contribute to the noise voltage. In general, beginning with Johnson's first work, it became clear that the noise voltage due to an impedance Z was due to the real part of that impedance. It is customary now to speak of the "dissipative" part of an impedance. The reactive components contribute no thermal noise voltage. Hence, in the Nyquist theorem Eq. (1) it is customary practice to write $\mathrm{Re}[Z(\nu)]$ instead of resistance R so that it becomes $S(\nu)d\nu = 4kT\mathrm{Re}[Z(\nu)]d\nu$, where $\mathrm{Re}[Z(\nu)]$ means the real part of $Z(\nu)$.

Nyquist's derivation for the thermal output of a resistor is based on a one-dimensional Rayleigh-Jeans analysis. The lossless transmission line connecting the two resistors is really a one-dimensional black-body cavity. The Rayleigh-Jeans formulation for a three-dimensional black-body cavity, it may be recalled, failed to give correctly the law for black-body radiation because it diverged at high frequencies in what came to be known as the "ultra-violet catastrophe".

The trouble was that the three-dimensional continuum could be subdivided endlessly to give an overwhelming preponderance of high-frequency oscillations, and this same difficulty pertains to the one-dimensional black body. The integral over all frequencies for the total noise power diverges if the resistor is pure ohmic and lacks reactance. Nyquist was fully aware of this, and pointed out that at high frequencies the equipartition value for the mean energy of a simple harmonic oscillator kT is to be replaced by the Planck value for the mean energy of the quantized oscillator at frequency ν:

$$E(h\nu) = \tfrac{1}{2}\,[h\nu \coth\,(h\nu/2kT)] \tag{2}$$

or:

$$h\nu = [\exp(h\nu/kT) - 1]^{-1} + \tfrac{1}{2}.$$

This substitution is cogent only at very high frequencies; in fact, it applies at frequencies beyond the range of detection of current practical electronic devices for noise measurements. A natural question is: why does the Rayleigh-Jeans method succeed in the one-dimensional case when applied to a supposedly simple resistor and fail when applied to black-body emissivity?

There are two answers to this question. The first is that actual resistors are not purely ohmic at all frequencies. The second is that the Rayleigh-Jeans' procedure does not introduce the reactances that must be taken into account when charge carriers are in motion. The admittance of the system determines the spectrum of charge-carrier velocities that can appear as fluctuations.

Concerning the nature of an ohmic resistor, the simplest approximation found in real resistors is that the resistor has, in effect, a shunt capacitance across it. The effect of the capacitance is to impart a frequency dependence to the resistor such that the integral over all frequencies converges. Depending on the construction of the resistor, it may also have hidden inductance. If one considers that the resistor has only a shunt capacitance C across it, the equipartition theorem leads to a value for the mean-square total noise voltage equal to kT/C. This is independent of resistance R and provides a macroscopic illustration of the role of admittance in determining noise magnitude. The hidden reactive components in real resistors determine the range of frequencies they may accommodate as black-body cavities.

2. Specific Models

CONCERNING the second answer given above, namely, the failure of a Rayleigh-Jeans type of analysis to introduce the necessary reactances for charge carriers in motion, there is a need for a specific model wherein the coupling between radiation field and charge carriers is indicated. Specific physical models for the role of electrons in the production of noise have been proposed. For example, J. Bernamont [3] (1937) and later D. A. Bell [4] (1938) derived the Nyquist theorem using the Drude model for an electron gas in a metal. The picture is amazingly simple: the electrons are independent of one another, they all have a common constant velocity v, a constant mean free path, and a constant time between collisions θ. The fluctuation current resulting from one electron is a series of flat-topped step functions whose time average is zero. A Fourier analysis for the spectral density of such a current leads to Eq. (1) if it is assumed that the electron kinetic energy is the equipartition value: $\frac{1}{2}mv^2 = \frac{3}{2}kT$, and the resistance is the classic value according to the Drude model: $R = (mL^2)/(e^2N\theta)$, where L is the length of conductor containing N independent electrons.

While this picture of the source of electrical noise leads to a Nyquist theorem, it has deficiencies that were pointed out by Lawson and Uhlenbeck [5] (1950). It does not provide a mechanism to prove that the distribution of amplitudes of voltage fluctuations is Gaussian. Moreover, it provides no basis for predicting whether or not there should be a change of noise when a current flows in a metallic resistor. The experimental data indicate that thermal noise is independent of current in metallic resistors, but that in nonmetallic resistors there is an excess noise introduced by a current. At the time, Lawson and Uhlenbeck stated that a satisfactory kinetic derivation of Nyquist's theorem had to be formulated, and went on to suggest that the most general theory should be applicable to an arbitrary system of bodies. It would require the use of Maxwell's equation in a statistical theory for bodies in equilibrium with the radiation field.

One way of understanding the curiously successful result provided by the Drude model is that it assumed no reactive components. The admittance of such a simple gas of electrons is the dc conductance which is essentially constant at least up to frequencies of the order of the reciprocal of the collision time θ, in the vicinity 10^{14} cps. A system with such an admittance could be expected to reproduce faithfully a step function current. It will be recalled that the Fourier transform for the delta function yields a constant spectral intensity. Nyquist's theorem as shown in Eq. (1) says that the spectral intensity for Johnson noise is constant over all frequencies. Hence, the assumption that the current due to random displacements of an electron is representable by a series of step functions tacitly bypasses the admittance problem (or the impedance problem, if one considers voltage fluctuations). It automatically contains the constant spectral intensity required.

In view of more recent remarkable and sophisticated developments in noise theory, a discussion of the Drude model may seem somewhat specious. It does, however, serve to point up the basic problem of the admittance as well as to give some historical background. With the advantage of hindsight, we can discern also the gradual development of the admittance concept as another illustration of the finite velocity of thinking; or perhaps one should call it the rms velocity of thinking. The basic problem in calculating noise has been the description of impedance at the microscopic level and its relation to the macroscopically observable impedance. Another way of stating the problem is: how is the mobility of charge carriers related to noise?

Thus, in the simplest model, an electron subjected to a random transient force would not give rise to a step-function current even if the force was a step function. The neighboring charges and their reactions would cause the given electron to evince a smeared out stepfunction response in its motion. The broad problem, then, in terms of charge carriers, rather than electrons in metallic conductors only, is to specify the frequency response of the motion of the charges and relate it to the macroscopic admittance.

This requires an accounting of the velocity distribution of the charges in thermal equilibrium, which is accomplished usually by assuming that the behavior of the resistor is not greatly modified by the passage of a small current. The distribution of thermal velocities is not changed significantly by the presence of a drift current because thermal velocities are far greater than the drift velocities. In short, the passage of a small current allows one to find the admittance of a system while it is essentially in a state of thermal equilibrium.

Indirectly related to the determination of admittance is a calculation of self-inductance of the aggregate of electrons in a metallic conductor by Brillouin[6] (1934). The Fermi-Dirac distribution of electron energies is used in the calculation of the fluctuations ΔI comprising current noise. One sums the magnetic interactions between electrons and thereby arrives at a value for the self-inductance of a system of N electrons in a conductor of length L and cross-sectional area, ΔA: $L = mL^2/Ne^2(\Delta A)^2$. Note that this resembles the value for resistance R in the Drude model with the $\theta/2$ replaced by $(\Delta A)^2$. The total current fluctuation in Brillouin's model is:

$$\langle \Delta I^2 \rangle = kTNe^2(\Delta A)^2/mL^2$$

or:

$$\langle \Delta I^2 \rangle = kT/L. \tag{3}$$

(Carets mean average value.) Equation (3) is also the result one obtains for the total current fluctuation from Nyquist's theorem based upon thermodynamic arguments.

The Brillouin calculation is intriguing in that it shows that reactances inside a conductor may be related to particle properties. Yet it has to be remembered that the unretarded magnetic-vector potential used by Brillouin was valid only for wavelengths large relative to the cross-sectional dimensions of the conductor, which makes the result of limited generality. The line of reasoning is suggestive and provocative, but has never been exploited and extended beyond what one might say is an early stage of its development.

3. Fluctuations and Transients

If a linear electrical network is given an input of the Heaviside step-function kind, the transient response gives the experienced observer a measure of the network. The pulse input contains frequency components at constant intensity over the entire range of frequencies. The network has an admittance which is frequency dependent and will therefore select a range of frequencies from the uniform spectrum available in the pulse. On a macroscopic scale the transient response of the system is unchanged as the step pulse is decreased in amplitude.

As the pulse becomes very small and microscopic, and the dimensions of the conductor or resistor become very large compared to the distances covered by thermal notions of charge carriers, what can one say about the so-called "transient response"? It should be noted that there has been no uniform nomenclature for the myriad current, or voltage, disturbances that culminate in the signal called noise. One finds the terms fluctuations, transients, pulses, and regression of a fluctuation used by different workers.

One essential fact that can be inferred about the submicroscopic fluctuations is that they contain steep wave fronts. The larger observable fluctuations into which they merge have the observed high-frequency components. To some extent there is a filtering effect in the coalescence of submicroscopic fluctuations into larger, macroscopic fluctuations in a random process. The filtering, in general, will be to reduce the very high-frequency components. Thus, the transition from the submicroscopic to the observable level is blurred.

A general method for dealing with the stochastic process which makes the observed macroscopic fluctuations is that of Langevin[7] (1908): one assumes that fluctuations in a variable such as current are due to random driving forces of a fictitious nature. Since they are fictitious, their magnitude and frequency spectrum are adjustable as needed. The Langevin procedure may be said to be heuristic and will be useful until more information about the submicroscopic nature of fluctuations shows how they are related to the observable fluctuations.

Interesting concepts of the fluctuations are given by Callen and Greene[8] (1952) and also by J. M. Richardson[9] (1955). We discuss here the latter's use of the idea of transients as given in his study of noise in driven systems. He defines a transient as the response to a physical force in the common usage of network theory. A voltage pulse applied to a circuit, or element thereof, gives rise to a transient response. But he also defines as a transient that which is the result of observation of a system beginning with certain initial conditions. This is called a transient response produced by selection: no physical force is applied. Thus, with regard to a resistor in thermal equilibrium, the observed transients, namely those produced by selection (no physical force has been brought to bear on the system), usually tend toward the equilibrium state. His second concept of transient then is simply the manner in which thermal fluctuations decay back to the equilibrium state.

Richardson's very general approach led to the conclusion that the transient response to a force, i.e., a physical perturbation resulting from a pulse, for example, was proportional to the transient response produced by selection. Since the latter is proportional to the noise, then, so also is the former. In circuit parlance this means that the pulse response is a kind of measure of the noise as given by Nyquist's theorem. It has to be emphasized that this result applies to a system in thermal equilibrium such as one might have if two equal resistors were connected by a transmission line and a voltage pulse were applied across one resistor.

The concepts of transient response as used by Richardson led to an important conclusion concerning driven systems. A driven system being, for example, a resistor connected to a dc source such as a battery. In a first approximation the resistor with current flowing is in a nonequilibrium steady state. Richardson showed that the transient response arising from a physical perturbation of such a system has no relationship to the transient response of the system produced by selection. Fluctuations in the driven system have a transient response which is independent of and logically distinct from the perturbation response. This result indicated that the Nyquist theorem as derived for an equilibrium system would not apply to one in a nonequilibrium steady state.

4. The Markoffian System

RICHARDSON'S conclusion that the Nyquist theorem might not apply to the noise in a driven system is based on a very general analysis. It assumed that the system in a state of thermal equilibrium, and in the driven state, could be described by an appropriate Hamiltonian for each state. The generality of Richardson's method is reminiscent of Nyquist's generality; the latter used thermodynamic reasoning while the former examined the various systems' impedance properties and their effect on transient responses. Also, while Richardson's conclusion was essentially correct, his procedure gave no concrete clue about how the noise would be modified in driven systems; no specific relation to mobility or involvement of relaxation mechanism is indicated.

Considerable clarification of the basic physics of driven systems has been provided by Melvin Lax[10] (1960) in his review of fluctuations from nonequilibrium steady states. In addition to reviewing the subject of noise, he crystallizes much of the thinking of recent years and also presents important generalizations in irreversible thermodynamics.

Lax's major assumption is that noise in an irreversible system is produced by a series of stochastic events of the Markoff type. The system is therefore called a Markoffian system, the state of which depends on the immediately preceding state but not beyond that state into the past. This is a first-order dependence of the probability of a state on the preceding history of the system. A second assumption is that all transition probabilities are time independent. Hence all displacements are time independent.

In a Markoffian system the present separates the future from the past. The probability of states in the future are conditional upon the present state of the system. Information about states attained in the past is not needed and, as Lax adds, is irrelevant when available. The bearing of this concept of a stochastic process on the physical picture was described in 1953 by Onsager and Machlup.[11] They pointed out that a system must be Markoffian if its future depends on its initial state, regardless of how it came to that initial state—whether by a spontaneous fluctuation, or a constraint, or an applied force.

Lax carries out a number of examples of calculations of noise in semiconductors where the charge-carrier-density fluctuations provide a good illustration of the Markoffian process. The number of charge carriers present at the end of a time interval Δt will depend on the number present at the beginning of Δt, as well as on the recombination rate. The number present at the end of Δt is independent of the previous history of the numbers of carriers up to the beginning of Δt.

In order to discuss the microscopic upheaval which we call a fluctuation, Lax uses an interesting phrase which is both apt as well as new in noise theory. The transient decay which follows upon a fluctuation he calls the regression of a fluctuation. In a material having a simple relaxation time θ, Lax's phenomenological description has the fluctuation transients decaying exponentially, $\exp(-t/\theta)$ back to equilibrium. This is a judiciously chosen model for the regression of a fluctuation. Lax points out that the "true regression" does not go back to the origin with a finite slope, but rather forms a cusp such that the slope is zero at $t=0$. The difference between the negative exponential and the so-called "true regression" near the origin $t=0$ provides a measure of the "forgetting time" of the Markoffian system. The break between the past and the present occurs in time intervals θ_f which are much smaller than the relaxation time θ.

This special property of Markoffian systems in forgetting the past permits Lax to make a far-reaching inference about the Nyquist theorem in driven systems, namely, that a Nyquist theorem always exists for a Markoffian system. It turns out to be a modified form of the theorem in that it has a correction factor. A Nyquist theorem will pertain because a Markoffian system does not distinguish between the different physical processes that brought it to its state at time $t=0$. An initial state of zero-velocity fluctuation at $t=0$ could have come about as a result of a spontaneous fluctuation or alternatively of a force which has a finite value for $t<0$ and is zero for $t>0$. The force may be calculated to produce the initial state of zero velocity at $t=0$, and the response of the system to this force in terms of the admittance leads to a Nyquist theorem. Thus, displacements independent of time and the isolation of the past from the future are properties of a physical system in which a kind of Nyquist theorem is always valid.

Lax shows that the spectral density in Eq. (1) may be written in one general form as

$$S(\nu) = 4kTRC, \tag{4}$$

where C is the modifying factor which has a variety of interpretations depending on the system under consideration.

5. The Modified Nyquist Theorem

LAX derives the Nyquist theorem in many different ways with various degrees of generality. In a particularly neat derivation he states that the Nyquist theorem relates the noise associated with velocity fluctuation at frequency ν to the admittance Y of the system and shows that for an equilibrium state:

$$S(\omega) = 4kT\text{Re}[Y(\omega)]C, \tag{5}$$

where

$$C = \langle q_i^2 \rangle_{QM} / \langle q_i^2 \rangle_{CL} \tag{6}$$

(Carets designate average.) C is the ratio of mean square of vibration amplitudes in quantum mechanical terms to that in classical terms, therefore, the numerator is from Eq. (2) above; $E(\hbar\omega) = (\hbar\omega/2)$ coth $(\hbar\omega/2kT)$, and the denominator is kT. Hence C becomes the universal correction factor: $C = (\hbar\omega/2kT)$ coth $(\hbar\omega/2kT)$ which is unity for $\hbar\omega \ll kT$ and becomes $\hbar\omega/2kT$ for $\hbar\omega \gg kT$. When the system is driven from equilibrium, C is the correction for the mean energy which has to be determined for the system to modify the Nyquist theorem.

If the system is considered only for the range of frequencies appropriate to the classical, nonquantum mechanical description Eq. (5) can be written as:

$$S(v,\omega) = 4m\langle vv \rangle \text{Re}[Y(\omega)], \tag{7}$$

where $Y(\omega) = \mu(\omega)/e$ is the admittance in terms of the mobility $\mu(\omega)$. At zero frequency the noise is related to the diffusion constant

$$D = m \langle vv \rangle \, \mu(o)/e, \tag{8}$$

where the mobility μ at zero frequency is the dc admittance. The Einstein [12] (1905) relation for diffusion constant is usually written as

$$D = kT(\mu/e) = kTL, \tag{9}$$

where $L = v/eE$ is the dc admittance to the applied force eE, since $v = \mu E$ is the definition of mobility μ. Equation (7) is one of several general forms of the Nyquist theorem. In it kT has been replaced by $m\langle vv \rangle$. In thermal equilibrium the kinetic theory value of $m\langle vv \rangle$ is a mean square value for kinetic energy and is kT. The general Eq. (7) is valid for a Markoffian system in which the regression of a fluctuation decays exponentially, with the single relaxation constant θ of the system in the exponent: $v(t) = \exp(-t/\theta) v(o)$. In a nonequilibrium steady state the distribution of velocities is such that $m \langle vv \rangle$ no longer equals kT. This factor occurs in also the Einstein relation [Eq. (9)] which should be thought of as being multiplied by a correction factor $C = \langle mv^2 \rangle / kT$, where the numerator in carets is an average that has to be determined by circumstances attending the nonequilibrium steady state.

The modification of distribution of velocities in the driven steady state is shown by Lax to lead to the lack of a relation between noise and admittance in the sense that Richardson had argued. The equilibrium distribution of velocities f_o is of the Boltzmann form and obeys the basic relation

$$\partial f_o / \partial v = -(mv/kT) f_o. \tag{10}$$

The left-hand side of this equation is a weight factor in the determination of the mobility $\mu(\omega)$. On the other hand, the right-hand side is a weight factor determining noise. The right-hand side of Eq. (10) contains vf_o which is logically distinct and different from $\partial f_o/\partial v$. If the distribution f_o is modified, when equilibrium conditions are changed, Eq. (10) no longer holds. Hence the proportionality between noise and admittance $Y(\omega) = \mu(\omega)/e$ does not necessarily exist. This amounts to saying that the special properties of thermal equilibrium distribution of velocities lead to a relation between noise and mobility; this relation being known as the Nyquist theorem. In driven systems the proportionality between $\partial f_o/\partial v$ and vf_o as shown by Eq. (10) no longer exists.

This is a satisfying result in that it directs one's thinking to the changes in the distribution of velocities. In a more general development Lax shows that the correction factor C may be: $C = S\langle \alpha\alpha \rangle / k$, where $\langle \alpha\alpha \rangle$ is a fluctuation in the variable α, S is an entropy, and k Boltzmann's constant. The classic energy term, kT, no longer appears in the denominator. Lax also shows that the fluctuation $\langle \alpha\alpha \rangle$ is proportional to kT and to a dc admittance in general. In a nonequilibrium state the proportionality is lost and the correction factor C in terms of entropy above has to be introduced.

Although there is progress in the basic concepts of noise in driven systems it is not yet possible, for example, to speculate that there may be new ways of defining temperature by noise in nonequilibrium states. Noise in resistors has been used to measure temperature in equilibrium situations. However, the nonequilibrium case has yet to be worked out in detail. Lax indicates the difficulties by means of an example. An impedance z is connected to another impedance Z, and z may be thought of as a load on Z. In thermal equilibrium it is possible to calculate the noise in z as well as in Z. In nonthermal equilibrium the calculation is not yet feasible: if the load impedance z has a real part, namely a dissipative part, it appears to modify the source impedance Z.

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