the MATHEMATICS of QUANTUM THEORY

By **G. Luzzatto** *and* **A. Wightman**

FROM July 21 to August 9, 1958, a summer course on Mathematical Problems of Quantum Theory of Particles and Fields was held in Varenna, Italy. The lectures took place in the Villa Monastero, on the shores of beautiful Lake Como; the place is well known to the readers of *Physics Today* since in other issues (Dec. 1957, Jan. 1958) there have been reports about conferences held there, and about other courses of the International School of Physics. The school is organized by the Italian Physical Society and is sponsored by the Italian National Science Council, by the Italian Government, by some Italian universities, by various local representatives, and occasionally by other institutions. Each summer it gives a number of courses (recently three or four), on special topics in physics. These courses are neither meetings of experts, nor quick courses at the graduate-school level. The goal is to put the persons with most experience in a field in touch with the younger or less experienced, in order to give the latter an up-to-date picture of recent research and open problems. The courses are organized quite independently and vary in length from a few days to three weeks in the present case.

The title of the course recalls that of the Lille Conference (Colloque sur les Problèmes mathématiques de la Théorie quantique des Champs, June 1957), and probably the similarity is not chance; they both were attempts to create a communication channel between field theorists and mathematicians. The director of the course, A. Borsellino of Genoa, firmly believes that such contacts are essential to reach a better understanding of field theory, both in its foundations and in its mathematical techniques; this is proved by the choice of the subjects and of the lecturers.

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The schedule at the school was adapted to the languor of the summer climate. The students and faculty assembled at 8:30 in the morning at the Villa Monastero, having come from various hotels and pensions in the town. From then until noon there were lectures with a half-hour break for coffee at 10:00. At 12:00 there was swimming in the lake. (A flight of steps leads directly down to the water in front of the Villa: so it was easy to make the transition from the turgid depths of quantum field theory to the cool, clear waters of Lake Como.) Lunch was at one. In the afternoon there was tea at 4:30, and with increasing frequency toward the end of the course, seminars at 5:00 (and sometimes even at 3:00).

THE first half of the course was mainly directed to the discussion of mathematical tools. Two outstanding mathematicians, L. Gårding (Lund) and J. L. Lions (Nancy), gave a general survey on the ideas and the methods of modern functional analysis. Gårding introduced various abstract spaces and concepts related to them (functions, seminorms, topologies; completeness, duality of linear spaces). In this framework, L. Schwartz's distribution theory was pre-

Participants in the course on Mathematical Problems of Quantum Theory of Particles and Fields posed for group photograph taken in the garden of the Villa Monastero. W. Pauli and W. Heisenberg are easily identifiable in second row.

sented. Lions discussed Fourier transforms, convolutions, Laplace transforms in distribution theory; he sketched the idea of vector-valued distributions and gave a complete exposition and proof of L. Schwartz's "nuclear theorem". A particular application of these general concepts, the theory of Lorentz invariant distributions and their Fourier transforms, was discussed by Gårding.

In this part of the course the idea was to acquaint listeners with the main notions and general framework of modern functional analysis, not to provide them a series of theorems which would make the solutions of the main problems of quantum field theory a purely technical exercise. The result was very potent and stimulating, and (at one time or another to nearly all physicists present) an overwhelming treatment of functional analysis.

Other mathematical ideas, for example, the foundations of group theory and the definitions of Grassmann and Clifford algebras, were developed only for immediate physical applications, and therefore mainly by the same physicists who discussed these applications. L. Michel (Lille) introduced the concepts of groups and their representations; G. Racah (Jerusalem) stated the basic results of the theory of Lie groups; general features of the structure, the representations, and the Lie algebra of the Lorentz group were given by Michel. Although rather standard, this part provided the necessary introduction to the following one. At this stage, Racah also gave a lecture on the "four-dimensional orthogonal groups": a little-known approach by Einstein (1931) leading to a classification of their representations.

The Wignerian analysis of relativistic invariance in quantum mechanics was explained by A. S. Wightman (Princeton). The physical discussion led to the mathematical problem of classifying the unitarily inequivalent representations of the inhomogeneous Lorentz group; these were described in some detail. The same formalism was also used by Michel to give a neat covariant treatment of the polarization of particles of arbitrary spin.

IN the second part of the course, G. Källén (Copenhagen) reviewed some main results obtained in actual field-theoretical calculations, emphasizing those points which seem to be particularly meaningful with regard to the general scheme. The topics discussed included the Bohr-Rosenfeld paper on the measurability of fields, the relation between Tamm-Dancoff and "incoming" states (a relation explicitly displayed for a model of a scalar field interacting with a classical external source), and calculations involving differences between infinite quantities (worked out in the vacuum polarization problem). The idea and the properties of renormalization procedures were illustrated by an analysis of the Lee model. For quantum electrodynamics, the infinities occurring in the perturbation series were considered, and various speculations on the high-energy behavior of matrix elements discussed.

R. Haag (Princeton) spoke about the general framework of quantum field theory, considering in detail the case of spin zero interacting fields. Formal collision theory, and mainly the relation of the interacting fields to the "incoming" and "outgoing" ones, was first discussed. One fact was particularly stressed: the representation of the Lorentz group, according to which a system of field transforms is determined only by its asymptotic states, i.e., is independent of the particular properties of the interaction. A model—neutral scalar field $A(x)$ with an interaction $A^3(x)$—was examined to see in it the implications arising from general requirements; finally, a formulation and a physical interpretation of the ideas of "asymptotic condition" and of "localized states" were given. As a result of this discussion it was clear that the "postulates" of an axiomatic approach to field theory ought to contain explicit properties of the spectrum, and not merely the condition of nonnegative energy states and the asymptotic condition in space, to be consistent with the standard physical interpretation.

H. Lehmann (Hamburg) gave explicit expressions for scattering matrix elements in terms of time ordered and in terms of retarded products; then he discussed the integral representation of these retarded products (or commutators). From the mathematical point of view, the same "Jost-Lehmann Dyson representation formula" was examined by Gårding. Using such tools, Lehmann derived the dispersion relations for a neutral scalar-charged scalar boson scattering process.

E. R. Caianiello (Naples) used Clifford and Grassmann algebras, and related notions (pfaffians, hafnians) to give a compact notation for the quantities appearing in quantum electrodynamics, and particularly in the Dyson expansion of the S-matrix. The Wick ordering theorem becomes in this framework a simple identity involving Clifford and Grassmann products; expressions are easily written, which automatically contain the sum of all n^{th} order graphs. Relations were derived involving the kernels of different order and their derivatives with respect to the coupling constant. Some results obtained in renormalizing the theory, by means of a definite prescription in the calculation of divergent integrals, were stated.

Wightman described how the requirement of a physically reasonable energy momentum spectrum gives rise to analyticity properties in x-space of the vacuum expectation values of products of field operators. He described how Hall extended this analyticity domain using relativistic invariance, and gave Jost's proof of the PCT theorem.

W. Heisenberg (Göttingen) discussed some aspects of the Tamm-Dancoff method and some features occurring in the Lee model (indefinite metric; ghosts; structure of the commutators) with the purpose of studying the nonlinear general field theory $\gamma_\mu \partial_\mu \psi \pm e^2 \gamma_\mu \gamma_5 \psi (\bar{\psi} \gamma_\mu \gamma_5 \psi) = 0$. Conjectures were made about the behavior of the solutions of such an equation; its invariance with respect to the Lorentz group, to the Pauli isospin and baryspin groups, to space inversion, time inversion, and charge conjugation was displayed. The Tamm-Dancoff method leads to two kinds of states which suggest an interpretation in terms of nucleons and pions; properties of these states were examined.

The lectures presented a fair picture of the present state of research in the foundations of quantum field theory with the exception of the subject of functional integration.

IN addition to these "courses", or series of lectures, a number of seminars on related subjects were given; the one by Gårding recalled above, one by M. Goldhaber (Brookhaven National Laboratory) on the situation in the study of the β-decay problem, two by J. G. Valatin (Birmingham) on the formalisms used in the theory of superconductivity, one by M. Froissart (CERN, Geneva) about "ghosts" in a relativistic field theory. But particularly we will all remember the two seminars which were, to the best of our knowledge, W. Pauli's last public speeches. He reported about the relation between spin and statistics, and about Gürsey's work on the group structure of the elementary particles; further remarks were added to this second talk by B. Touschek (Rome), who discussed the symmetry properties of an assembly of Majorana fields (a generalization of the Pauli group). Pauli was also chairman for an informal summary discussion on the last morning of the school: various questions which had arisen during the three weeks were answered by the lecturers and an opportunity was given for remarks and conjectures.

The presence of approximately 30 "listeners", in addition to the 51 "students" and the 14 "lecturers", indicates that the enterprise aroused considerable interest. The organizers were forced to reject an almost equal number of applications, as the room in which the classes are held is small, and also because too large an audience would make it more difficult to reach the goal of close contact between speakers and listeners.

The city of Varenna (which provided a reception and fireworks display), the city of Como (which provided a boat trip and a magnificent dinner en route), the organizers (who arranged a performance of a play by Metastasio in the garden of the Villa), and the countryside conspired to make the school a very pleasant occasion for all involved. In the opinion of the present authors it was also a success scientifically.