Boundary Conditions on Fast Oscillation and Pivot Oscillation Speeds in *n*-Chain Kapitza Pendulum Systems

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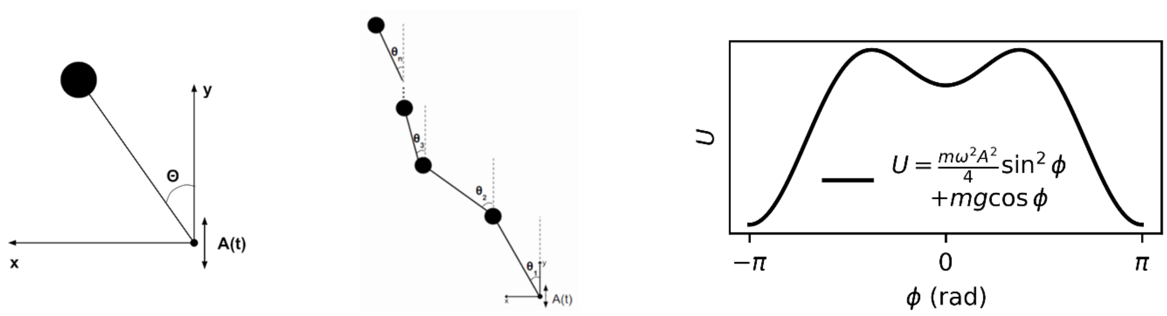
Abstract. This study investigates the stability conditions for an n-chain Kapitza pendulum system. A single Kapitza pendulum has an interesting property in that its stable equilibrium position is the upright position. The motion of the pendulum can be separated into a small-yet-fast component and a large-yet-slow component. This property extends to the n-chain system. This study, through the separation of scales technique, confirms that the small, fast component does not diverge with n and finds the criteria for the stability of the nth pendulum. In doing so, this study also finds an upper bound for the sum of the fast components. Additionally, numerical simulations of the n-chain Kapitza system showed a trend of increasing unstable states with n.

INTRODUCTION

An interesting variation of the classical pendulum, a single Kapitza pendulum is a system that consists of a typical pendulum setup with a pivot that oscillates vertically by a function $A(t)$. This vertical oscillation, when rapid enough, causes the stable equilibrium of the system to switch to the upright position, with the downward hanging position switching to the unstable equilibrium position, as when $\theta = 0$ in Fig. 1(a). This phenomenon can be analyzed algebraically by separating the angular displacement of the pendulum from the vertical, denoted as θ, into a fast-yet-small component δ and a slow-yet-large component ϕ, as in [1]. This separation of scale can be observed in [2] and [3]. Reference [5] shows a two-chain Kapitza pendulum where the fast oscillations seem faster for the top pendulum. Extrapolating this to a general n-chain Kapitza pendulum, we can conjecture whether the fast oscillations of an n-chain Kapitza pendulum will grow, shrink, or stay constant with n. Also, within an n-chain Kapitza pendulum, we can wonder how additional joints in the form of a higher n value will affect the pivot oscillation ω needed to keep the system at a stable equilibrium.



(a) (b) (c)

FIGURE 1. The coordinate system and setup of variables for (a) a single Kapitza pendulum and (b) an n-chain Kapitza pendulum. (c) The potential energy graph for a single Kapitza pendulum.

THEORETICAL MODEL

For a single Kapitza pendulum, as defined in Fig. 1(a), we can define its position vector as $\vec{r} = (l\sin\theta,\ l\cos\theta + A)$, where $A = a\cos\omega t$, the oscillatory function of the pivot. Then, the Lagrangian is

$$L = K - U = \frac{1}{2}m\left(l^2\dot{\theta}^2 - 2\dot{A}l\dot{\theta}\sin\theta + \dot{A}^2\right) - mg(A + l\cos\theta). \tag{1}$$

As the pivot oscillation function $A(t)$ is given by the system, the only free variables are θ and $\dot{\theta}$. After substituting Eq. (1) into the Euler-Lagrange equation, we get the following equation of motion:

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = l\ddot{\theta} - \ddot{A}\sin\theta - g\sin\theta = 0\,. \tag{2}$$

Following Kapitza's insight in [1], we try a solution of $\theta(t) = \phi(t) + \delta(t)$, where $\delta(t)$ is assumed to be a fast-yet-small oscillation and $\phi(t)$ is a slow-yet-large oscillation. We assume that δ is small enough that $\sin\delta = \delta$ and $\cos\delta = 1$. Essentially, we are decomposing $\theta(t)$ into two components that operate at different time scales and magnitudes. Substituting $\theta(t) = \phi(t) + \delta(t)$ into Eq. (2) and using trigonometric angle addition identities, we get the following ordinary differential equation (ODE):

$$\ddot{\phi} + \ddot{\delta} + \frac{\omega^2 a}{l}\cos\omega t\sin\phi + \frac{\omega^2 a}{l}\delta\cos\omega t\cos\phi - \frac{g}{l}\sin\phi - \frac{g}{l}\delta\cos\phi = 0. \tag{3}$$

Given the drastically different time scales at which δ and ϕ oscillate, we can assume that the angular frequency of δ is magnitudes faster than that of ϕ. Since ϕ and δ are sinusoidal functions, $\ddot{\delta}$ is, consequently, magnitudes larger than $\ddot{\phi}$. As in [6], when the torque of the pivot oscillation exceeds the torque of the gravitational force, we can assume $\omega^2 A \gg g$ as our condition for stabilization. In that case, the last two terms of Eq. (3) can be ignored. Given the extremely small scale of the fast oscillation δ, we can approximate $|\delta| \ll 1$. Consequently, we can ignore the fourth term in Eq. (3). This yields

$$\ddot{\delta} + \frac{\omega^2 a}{l}\cos\omega t\sin\phi = 0. \tag{4}$$

As we have assumed the angular frequency of δ is magnitudes faster than of ϕ, by analyzing Eq. (4) within the time scale that δ operates in, we can effectively treat ϕ as a constant. Integrating Eq. (4) twice with respect to t gives

$$\delta(t) \approx \frac{a}{l}\cos\omega t\sin\phi. \tag{5}$$

As we will investigate the stability conditions of the nth pendulum later in the paper, a graph of the potential energy function of the single Kapitza pendulum will suffice for now. According to [7], the effective potential of the system is as shown in Fig. 1(c). The potential is at a stable equilibrium when $\phi = 0$, meaning that unless the pendulum passes the unstable equilibrium points (the two peaks next to $\phi = 0$), the pendulum will return to the upright position.

In the general case of an *n*-chain Kapitza pendulum, the setup is as shown in Fig. 1(b), with each pendulum bob having a uniform mass m connected by a rigid rod with negligible mass of length l. θ_i is the angle of the ith pendulum with the vertical, and the pivot point is oscillating vertically by the function $A(t) = a\cos\omega t$. We define the origin as the position of the pivot point when $\cos\omega t = 0$ and $\vec{r_i}$ as the position of the ith pendulum bob. Then, $\vec{r_i}$ is

$$\vec{r_i} = \left(l\sum_{j=1}^{i}\sin\theta_j\,, l\sum_{j=1}^{i}\cos\theta_j + A\right). \tag{6}$$

From Eq. (6), we can find the kinetic and potential energies of the ith pendulum. We find the Lagrangian by summing the potential energies of all n pendulums and subtracting that from the sum of their kinetic energies. For the sake of clarity, we define a new variable $u_i \equiv (n + 1 - i)$. Simplifying the nested summations in the Lagrangian gives

$$\mathrm{L} = m\left(\sum_{i=1}^{n}\frac{1}{2}l^2\mathrm{u_i}\dot{\theta_i}^2 + l^2\sum_{i=2}^{n}\sum_{k=1}^{i-1}\mathrm{u_i}\cos(\theta_i - \theta_k)\,\dot{\theta}_i\dot{\theta}_k - l\dot{A}\sum_{j=1}^{n}\mathrm{u_j}\dot{\theta}_j\sin\theta_j + n\dot{A}^2\right) - \left(mlg\sum_{i=1}^{n}\mathrm{u_i}\cos\theta_i + \mathrm{m}gnA\right), \tag{7}$$

where the first line is the kinetic energy terms and the second line is the potential energy terms. We can get the equation of motion for the ith pendulum from the Euler-Lagrange equation:

$$(g+\ddot{A})u_i\sin\theta_i - l\left(\sum_{k=1}^{i-1} u_i\,\dot{\theta}_k^2\sin(\theta_i-\theta_k) + \sum_{k=i+1}^{n} u_k\,\dot{\theta}_k^2\sin(\theta_i-\theta_k)\right)$$

$$-l\left(u_i\ddot{\theta}_i + \sum_{k=1}^{i-1} u_i\ddot{\theta}_k\cos(\theta_i-\theta_k) + \sum_{k=i+1}^{n} u_k\ddot{\theta}_k\cos(\theta_i-\theta_k)\right)$$

$$+\,2l\sum_{k=i+1}^{n} u_k\,\dot{\theta}_i\,\dot{\theta}_k\sin(\theta_i-\theta_k) = 0. \tag{8}$$

Since $\cos(\theta_i-\theta_k)=1$ for $i,k=n$, we can tack a $\cos(\theta_i-\theta_k)$ term onto the $u_i\ddot{\theta}_i$ term in Eq. (8) and integrate it within the $\sum_{k=1}^{i-1} u_i\,\ddot{\theta}_k\cos(\theta_i-\theta_k)$ term to form $\sum_{k=1}^{i} u_i\,\ddot{\theta}_k\cos(\theta_i-\theta_k)$, a transformation only valid when $i=n$. Since we are interested in the stability of the nth pendulum, we let $i=n$ to get its equation of motion. Additionally, as nonsensical terms such as u_{n+1} are not properly defined in our summation, we take their value to be zero:

$$\sin\theta_n\,(g+\ddot{A}) - l\sum_{k=1}^{n}\ddot{\theta}_k\cos(\theta_n-\theta_k) - l\sum_{k=1}^{n}\dot{\theta}_k^{\,2}\sin(\theta_n-\theta_k) = 0. \tag{9}$$

As we did for θ in the single Kapitza pendulum case, we let $\theta_i \approx \phi_i + \delta_i$. The same assumptions that we applied to δ and ϕ in the single Kapitza pendulum case, we now apply to δ_i and ϕ_i. Simplify Eq. (9) for the fast component δ and applying the same conditions as we did to the single Kapitza pendulum case gives

$$\sum_{k=1}^{n}\delta_k\cos(\phi_n-\phi_k) \;= \frac{a}{l}\cos\omega t\sin\phi_n. \tag{10}$$

The maximum value of the left-hand side occurs when $\cos(\phi_n-\phi_k)=1$, and the maximum value of the right-hand side occurs when $\sin\phi_n = 1$. Using these maximal values, we obtain the upper bound for fast oscillations in an n-chain Kapitza pendulum system as

$$\sum_{k=1}^{n}\delta_k \le \frac{a}{l}\cos\omega t. \tag{11}$$

Now that we have found a restriction on δ, we turn our focus to the conditions under which ϕ_n is in stable equilibrium in the upright position. According to [1], δ for $n=1$ is proportional to $\cos\omega t$. As shown in Eq. (11), given that an arbitrary sum of δ_k is a multiple of $\cos\omega t$, we can make an ansatz that there exists an ϵ such that $\epsilon_k \equiv \delta_k/\cos\omega t$. Our ansatz also implies that δ is a sinusoidal function with a period of ω. We can average Eq. (9) over a time of $2\pi/\omega$ by taking its time integral from $t=0$ to $\frac{2\pi}{\omega}$. Since $\delta \propto \cos\omega t$, $\int_0^{\frac{2\pi}{\omega}}\delta_k\,dt = 0$ and $\int_0^{\frac{2\pi}{\omega}}\delta_k\cos\omega t = \frac{1}{2}\epsilon_k$. Applying the same three assumptions on δ and ϕ we used when solving the equation of motion for the single Kapitza pendulum gives the following equation:

$$\frac{g}{l}\sin\phi_n - \frac{a\omega^2}{2l}\epsilon_n\cos\phi_n = \ddot{\phi}_n + \sum_{k=1}^{n-1}\ddot{\phi}_k\cos(\phi_n-\phi_k) + \sum_{k=1}^{n}\dot{\phi}_n^{\,2}\sin(\phi_n-\phi_k). \tag{12}$$

To investigate the stability conditions, we must first derive an expression for the potential energy of the nth pendulum. We notice one of the terms in the identity $\ddot{\phi}_n = \frac{d}{d\phi_n}\left(\frac{1}{2}\,\dot{\phi}_n^2\right)$ resembles a kinetic energy term. Thus, we integrate Eq. (12) with respect to ϕ_n to get

$$\frac{g}{l}\cos\phi_n - \frac{a\omega^2}{2l}\left(\frac{a}{4l}\cos 2\phi_n + \sum_{k=1}^{n-1}\epsilon_k\sin\phi_n\right) + \frac{1}{2}\dot{\phi}_n^2 + \sum_{k=1}^{n-1}\ddot{\phi}_k\sin\,(\phi_n-\phi_k) - \sum_{k=1}^{n-1}\dot{\phi}_k^2\cos(\phi_n-\phi_k) \;= C, \tag{13}$$

where C is an integration constant. We notice that the term $\frac{1}{2}\dot{\phi}_n^2$ in Eq. (13) is proportional to a term for kinetic energy. Since the sum of the kinetic energy term and other terms is a constant, we can interpret Eq. (13) as a conservation of energy theorem for the nth Kapitza pendulum. Correspondingly, we define a potential energy term as

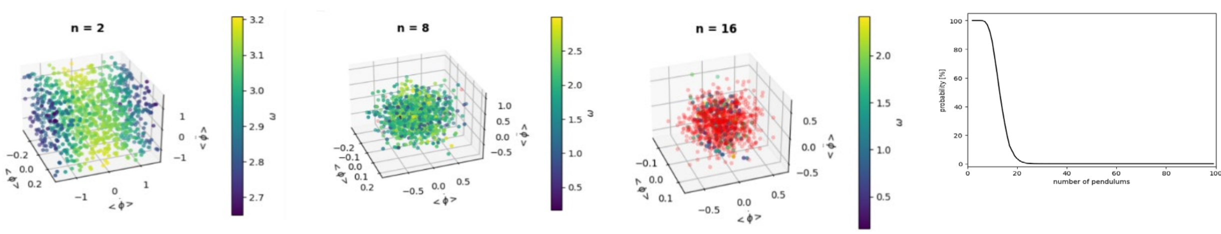
$$\frac{U}{ml^2} = \frac{g}{l}\cos\phi_n - \frac{a\omega^2}{2l}\left(\frac{a}{4l}\cos 2\phi_n + \sum_{k=1}^{n-1}\epsilon_k\sin\phi_n\right) + \sum_{k=1}^{n-1}\ddot{\phi}_k\sin(\phi_n-\phi_k) - \sum_{k=1}^{n-1}\dot{\phi}_k^2\cos(\phi_n-\phi_k). \tag{14}$$

We can determine whether the equilibrium point is stable by taking the second derivative of U with respect to ϕ_i. For this case, we want to find the stability condition when the system is close to being in a completely upright position. Mathematically, the system would be restricted to $\phi_n = 0$ and $\phi_k \cong 0$. We apply our conditions to Eq. (14) and solve for $\partial^2 U/\partial \phi_n^2 > 0$ to get the lower bound

$$\omega > \frac{1}{a}\sqrt{g - l\sum_{k=1}^{n-1}(\ddot{\phi}_k\phi_k + \dot{\phi}_k^{\,2})}. \tag{15}$$

RESULTS AND CONCLUSION

Given the condition that ϕ_k is around zero, we can run a numerical calculation where -0.34 rad $<\phi_k<$ 0.34 rad. For the velocities and accelerations, the respective ranges are $-\pi/2<\dot{\phi}_k<\pi/2$ and $-\pi/2<\ddot{\phi}_k<\pi/2$. For simplicity, we let $a = l = 1$ m. The minimum angular velocity of the pivot required for a stable equilibrium $\omega_{\min}$ is calculated from Eq. (15) for various n values, with 1,000 iterations run for each n to represent a wide range of possible configurations. The result is depicted in Figs. 2(a-c). While the numerical calculation randomly selected the values for ϕ_i, $\dot{\phi}_i$, and $\ddot{\phi}_i$, the graph's three axes show $\langle\phi\rangle$, $\langle\dot{\phi}\rangle$, and $\langle\ddot{\phi}\rangle$, which are the values averaged over all pendulums. The lower bound for ω is displayed in color; red means that no stable solution was found with such a configuration.



(a) (b) (c) (d)

FIGURE 2. Graphs of $\omega_{\min}$ for (a) $n = 2$, (b) $n = 8$, and (c) $n = 16$ (c). Red dots indicate that $\omega_{\min}$ is an imaginary number. (d) The probability of $\omega_{\min}$ having a real number solution.

In experimental configurations of the n-chain Kapitza pendulum system where $n \geq 2$, such as in [4], the topmost pendulum vibrates more vigorously. Still, there is an upper bound to the sum of all δ_i in an n-chain Kapitza system. What is surprising is that the upper bound in Eq. (11) is independent of n. This implies that all δ_i have the same phase but different magnitudes, which means the magnitudes of the individual fast oscillations change in reaction to n to obey the inequality in Eq. (11). The magnitude of all fast oscillations is distributed and increases as one goes up the system. Qualitatively, we can consider the inertia associated with the ith pendulum. The higher up the n-chain, the less mass a given pendulum must support. The differing inertias cause the variation in magnitude of δ_i.

We can see from Eq. (15) that for some configurations of ϕ, $\dot{\phi}$, and $\ddot{\phi}$, the lower bound for the pivot frequency is imaginary. We interpret the scenario to mean that no ω exists that could make the nth pendulum stable at the upright position. In Figs. 2(a-c), we observe that as n increases, more red dots appear. Since they represent unconditionally unstable states, Fig. 2 suggests that as n increases, the likelihood of a stable state decreases. Figure 2(c) shows this relationship between n and the probability of finding a stable state. We observe a sharp decline between $n = 10$ and $n = 20$, after which the probability is near zero. In Figs. 2(a) and (b), where n is 2 and 8, we can see that $\omega_{\min}$ decreases with distance from the origin. This can be explained by considering the stability of an n-chain system in terms of the effective potential of the top pendulum. The farther a state deviates from the origin, the more it contributes to making $\partial^2 U/\partial \phi_n^2$ positive. This means less contribution from ω is required for the system to be in a stable state.

As shown in Eqs. (11) and (15), the fast oscillation components of an n-chain Kapitza pendulum system have an upper bound that is independent of n and hence do not diverge as $n \to \infty$, and there exists a distribution of conditionally stable and absolutely unstable states for certain values of n.

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