The Laws of Reflection, Refraction, and Relative Amplitudes

Derived from Boundary Conditions of Maxwell's Equations

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The laws of reflection and refraction in geometrical optics, and the relative amplitudes of the incident, reflected, and refracted waves, follow as deductive consequences of boundary conditions that emerge from Maxwell's equations.[1]

When electric and magnetic fields encounter matter, the molecules typically become polarized—or if they were already polarized, their dipole moments are enhanced. These induced electric and magnetic dipole moments produce additional electric and magnetic fields of their own. Taking these polarization effects into account, the Maxwell equations that relate charges and currents to the fields can be written in terms of the electric field $\mathbf{D}$ due only to free (excess) charges of density $\rho_{\boldsymbol{o}}$ and the magnetic field $\mathbf{H}$ due only to free currents of density $\mathbf{j}_o$:

$$\nabla\cdot\mathbf{D} = \rho_{\boldsymbol{o}} \tag{1}$$

$$\nabla\times\mathbf{H} = \mathbf{j}_o + \frac{\partial\mathbf{D}}{\partial t}. \tag{2}$$

The Maxwell equations with no source terms are conveniently written in terms of the total electric field $\mathbf{E}$ and the total magnetic field $\mathbf{B}$:

$$\nabla\cdot\mathbf{B} = 0 \tag{3}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}. \tag{4}$$

The source-free and total fields are related to each other through parameters that encode the effects of polarization,[2] in the so-called constitutive relations, which in homogeneous and isotropic media[3] take the form

$$\mathbf{D} = \epsilon\mathbf{E}, \tag{5}$$

where ϵ is the material's permittivity, and

$$\mathbf{H} = \frac{\mathbf{B}}{\mu}, \tag{6}$$

where μ is the material's permeability. If the medium contains no matter at all—a vacuum—the permittivity of free space is $\epsilon_o = 8.85\times10^{-12}\ \mathrm{F/m}$, with permeability $\mu_o = 4\pi\times10^{-7}\ \mathrm{H/m}$. Significantly,

$$\mu_o\epsilon_o = \frac{1}{c^2}, \tag{7}$$

where c denotes the speed of light in vacuum, $c = 3\times10^8\ \mathrm{m/s}$. Equations for electromagnetic waves in matter readily follow from Maxwell's equations. For instance, taking the curl of Eq. (2) and using a vector identity yields

$$\nabla(\nabla\cdot\mathbf{H}) - \nabla^2\mathbf{H} = (\nabla\times\mathbf{j}_o) - \frac{\partial}{\partial t}(\nabla\times\mathbf{D}). \tag{8}$$

You can use Eqs. (3) and (6), eliminate the first term, and use Eqs. (4) and (5) to turn Eq. (8) into the inhomogeneous wave equation for $\mathbf{H}$,

$$\nabla^2\mathbf{H} - \mu\epsilon\frac{\partial^2\mathbf{H}}{\partial t^2} = -(\nabla\times\mathbf{j}_o), \tag{9}$$

showing the magnetic wave's speed to be $1/\sqrt{\mu\epsilon}$. A similar wave equation can be derived for $\mathbf{D}$ by taking the curl of Eq. (4), then using the constitutive relations and the other Maxwell equations to derive

$$\nabla^2\mathbf{D} - \mu\epsilon\frac{\partial^2\mathbf{D}}{\partial t^2} = \nabla\rho_o + \mu\epsilon\frac{\partial\mathbf{j}_o}{\partial t}. \tag{10}$$

Comparing Eqs. (9) and (10), we can see that electromagnetic waves in a medium travel at the speed

$$v = \frac{1}{\sqrt{\mu\epsilon}}. \tag{11}$$

Since the index of refraction n may be defined as $n = c/v$, it follows from Eq. (11) that

$$n = c\sqrt{\mu\epsilon}. \tag{12}$$

These are input facts for describing electric and magnetic fields in a material medium. Now we need the boundary conditions to see the effects on the fields when they pass from one medium into another.

Maxwell's Boundary Conditions

Consider two homogeneous, isotropic, nonconducting media that are in contact. Whatever the shape of the contact boundary, any infinitesimal section of it may be considered a plane. In what follows we assume that no free charges and no free currents exist on the boundary, so that $\rho_o = 0$ and $\mathbf{j}_o = 0$ there. In source-free regions, the integral versions of Eqs. (1)-(4) read

$$\oint \mathbf{D} \cdot \hat{\mathbf{n}}\, dA = 0 \tag{13}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = \frac{d}{dt}\int_S \mathbf{D} \cdot \hat{\mathbf{n}}\; dA \tag{14}$$

$$\oint \mathbf{B} \cdot \hat{\mathbf{n}}\, dA = 0 \tag{15}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \hat{\mathbf{n}}\; dA. \tag{16}$$

Equations (13) and (15) apply to any closed surface; in Eqs. (14) and (16) S is any nonclosed surface that has closed path C for its boundary.

Imagine a small, closed Gaussian surface, a wee little box that straddles the boundary between the two media.

Let the box's surfaces parallel to the boundary have area A. Let the box sides that connect the surfaces above and below the boundary have height h, which will go to zero, leaving the two parallel surfaces skimming the boundary on either side of it (Fig. 1). Equation (13) applied to this closed surface implies the normal component of $\mathbf{D}$ is continuous, $(D_{n1} - D_{n2})A = 0$, or $\Delta D_n = 0$ for short. Similarly, $\Delta B_n = 0$ from Eq. (15).

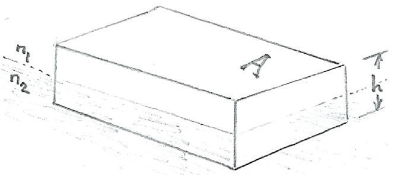

Figure 1. A Gaussian closed box used to derive $\Delta D_n = 0$ and $\Delta B_n = 0$.

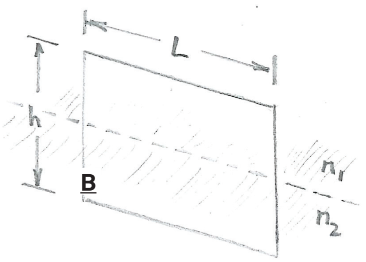

Figure 2. A closed path used to derive $\Delta E_t = 0$ and $\Delta H_t = 0$.

Consider next a small rectangular contour C in Eq. (16). Let two sides of C have length L and be parallel to the boundary, one side above and one side below. Let the other two sides of C have length h, which will go to zero, leaving the two sides of length L skimming the boundary on either side (Fig. 2). Equation (16) applied to this situation implies the tangential component of $\mathbf{E}$ is continuous, $(E_{t2} - E_{t1})L = 0$, or $\Delta E_t = 0$ for short. Similarly, Eq. (14) yields $\Delta H_t = 0$. To summarize, Maxwell's equations for these situations require

$$\Delta D_n = 0, \tag{17}$$

$$\Delta E_t = 0, \tag{18}$$

$$\Delta B_n = 0, \tag{19}$$

$$\Delta H_t = 0. \tag{20}$$

Now let the boundary of contact between the two media be designated as the xy plane, with the positive z-axis in medium 1. Let the refractive indices of the two media be $n_1 = c\sqrt{\mu_1\epsilon_1}$ for $z > 0$ and $n_2 = c\sqrt{\mu_2\epsilon_2}$ for $z < 0$. Consider a plane wave in medium 1 approaching the boundary with its corresponding ray, making the angle θ relative to the $+z$-axis (Fig. 3). Some of the energy carried by the electromagnetic wave will be reflected, and some energy will be refracted.[4] As shown in Fig. 4, let the reflected ray leave the boundary at angle θ^* from the normal, while the ray of the refracted wave enters medium 2 at the angle θ' as measured from the normal line.

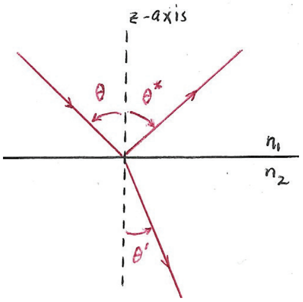

Figure 3. The angles made by the incoming (θ), reflected (θ^*), and refracted (θ') rays, relative to the normal of the boundary surface.

With a wave-number vector $\mathbf{k}$ describing the direction of propagation, let an incident harmonic wave be expressed as

$$\mathbf{E}_{inc}(\mathbf{r},t) = \mathbf{E}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{21}$$

the reflected wave as

$$\mathbf{E}_{refl}(\mathbf{r},t) = \mathbf{E}^* e^{i(\mathbf{k}^*\cdot\mathbf{r} - \omega^* t)}, \tag{22}$$

and the refracted wave as

$$\mathbf{E}_{refr}(\mathbf{r},t) = \mathbf{E}' e^{i(\mathbf{k}'\cdot\mathbf{r} - \omega' t)}. \tag{23}$$

For any of these fields, Eq. (4) requires, for their respective $\mathbf{k}$ and ω,

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}, \tag{24}$$

while Eqs. (1) and (3) give $\mathbf{k}\cdot\mathbf{D} = 0$ and $\mathbf{k}\cdot\mathbf{B} = 0$, respectively. We see, therefore, that the propagation vector **k**, the electric field, and the magnetic field are mutually perpendicular. This fact along with $v = \omega/k$ and Eqs. (24) and (12) imply, for each of the three waves,

$$|\mathbf{B}| = \sqrt{\mu\epsilon}\,|\mathbf{E}| = \frac{n}{c}|\mathbf{E}|. \tag{25}$$

Since we assume that no free charges or currents reside on the boundary surface between media 1 and 2, the normal and tangential component of the electric and magnetic fields must be continuous at $z = 0$ at any time and at any location on the boundary. Therefore, at the boundary, the phases of the incident, reflected, and refracted waves must be equal:

$$e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = e^{i(\mathbf{k}^*\cdot\mathbf{r} - \omega^* t)} = e^{i(\mathbf{k}'\cdot\mathbf{r} - \omega' t)}. \tag{26}$$

Requiring Eq. (26) to hold for any time t means

$$\omega = \omega^* = \omega', \tag{27}$$

so classical electrodynamics predicts that the reflected and refracted waves will emerge from the boundary interaction with unchanged frequencies, which seems validated in everyday life—if our mirror is made of uncolored glass, then our reflection comes back to us with the same color as the light incident on it. (This classical electrodynamics background illustrates the significance of the Compton effect.[5]) Using Eq. (27) in Eq. (26) implies that, for each wave with its respective parameters, $\omega = vk = ck/n$, so that

$$\frac{k}{n} = \frac{k^*}{n^*} = \frac{k'}{n'}, \tag{28}$$

where $n_1 = n = n^*$ and $n_2 = n'$. Also, in the boundary surface where $z = 0$, where the r vectors depend only on x and y, the continuity of the field components in homogeneous, isotropic media implies

$$\mathbf{k}\cdot\mathbf{r} = \mathbf{k}^*\cdot\mathbf{r}^* = \mathbf{k}'\cdot\mathbf{r}', \tag{29}$$

which implies

$$k\sin\theta = k^*\sin\theta^* = k'\sin\theta'. \tag{30}$$

Examine the left equality of Eq. (30),

$$k\sin\theta = k^*\sin\theta^*. \tag{31}$$

Since the incident and reflected wave are in the same medium so that $n = n^*$, Eq. (28) gives $k = k^*$, and Eq. (31) thereby requires

$$\theta = \theta^*, \tag{32}$$

the law of reflection. From the right equality of Eq. (30) we have

$$k\sin\theta = k'\sin\theta'. \tag{33}$$

Recalling $k = n\omega/c$ and Eq. (27), Eq. (33) becomes

$$n\sin\theta = n'\sin\theta', \tag{34}$$

the law of refraction, Snell's law.

We might say that Eq. (26) is the kinematic part of the boundary conditions in Maxwell's equations, because the laws of reflection and refraction describe the directions of the reflected and refracted rays relative to the incident ray. The dynamic part of the boundary conditions will tell us the amplitudes of the reflected and refracted waves relative to the incident one, which determines where the energy goes, since the energy carried by a wave is proportional to the square of its amplitude. These results are called the Fresnel equations.

The Fresnel Equations

Since electromagnetic radiation consists of transverse waves, the electric and magnetic fields can be polarized into two sets of orthogonal modes. Therefore, we consider two independent polarization states incident on the boundary:

(1) The incident $\mathbf{E}$ comes into the boundary tangent to the xy plane, with $\mathbf{B}$ in the xz plane; and

(2) The incident $\mathbf{B}$ comes into the boundary tangent to the xy plane and $\mathbf{E}$ lies in the xz plane.

Case 1: $\mathbf{E}$ tangent to the xy boundary plane, $\mathbf{B}$ parallel to the xz plane

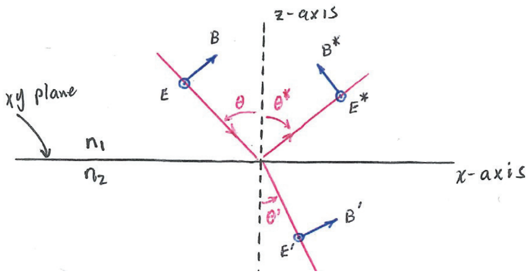

Figure 4. Case 1 polarization: $\mathbf{E}$ *tangent to the xy boundary plane,* $\mathbf{B}$ *parallel to the xz plane.*

The boundary condition $\Delta E_t = 0$ implies

$$E + E^* = E'. \tag{35}$$

The boundary condition $\Delta H_t = 0$ along with $\mathbf{H} = \mathbf{B}/\mu$ and $\theta = \theta^*$ yields

$$\frac{1}{\mu_1}(B - B^*)\cos\theta = \frac{1}{\mu_2}B'\cos\theta'. \tag{36}$$

By virtue of Eq. (25), this may be written

$$\frac{n}{\mu_1}(E - E^*)\cos\theta = \frac{n'}{\mu_2}E'\cos\theta'. \tag{37}$$

Equations (35) and (37) are two equations in two unknowns for E^*/E and E'/E. After a bit of algebra, this pair of equations gives the first set of Fresnel equations:

$$\frac{E^*}{E} = \frac{\frac{n}{\mu_1}\cos\theta - \frac{n'}{\mu_2}\cos\theta'}{\frac{n}{\mu_1}\cos\theta + \frac{n'}{\mu_2}\cos\theta'} \tag{38a}$$

and

$$\frac{E'}{E} = \frac{2\frac{n}{\mu_1}\cos\theta}{\frac{n}{\mu_1}\cos\theta + \frac{n'}{\mu_2}\cos\theta'}. \tag{38b}$$

If $\mu_1 \approx \mu_2$ (which is typical for nonmagnetized material), then these Fresnel equations for Case 1 ($\mathbf{E}$ tangent to the boundary) simplify to

$$\frac{E^*}{E} = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'} \tag{39a}$$

$$\frac{E'}{E} = \frac{2n\cos\theta}{n\cos\theta + n'\cos\theta'}. \tag{39b}$$

Case 2: $\mathbf{B}$ tangent to the xy boundary surface, $\mathbf{E}$ parallel to the xz plane

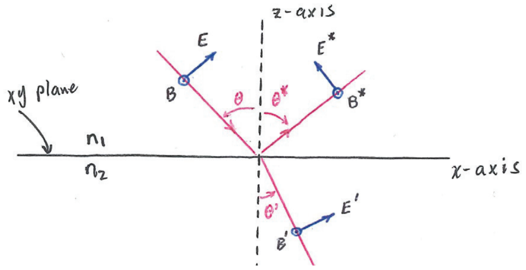

Figure 5. Case 2 polarization: $\mathbf{B}$ *tangent to the xy boundary plane,* $\mathbf{E}$ *parallel to the xz plane.*

The boundary condition $\Delta H_t = 0$ gives

$$H + H^* = H'. \tag{40}$$

Using $H = B/\mu = nE/c\mu$ and assuming $\mu_1 = \mu_2$ turns Eq. (49) into

$$n(E + E^*) = n'E'. \tag{41}$$

The boundary condition $\Delta E_t = 0$ implies

$$(E - E^*)\cos\theta = E'\cos\theta'. \tag{42}$$

Now Eqs. (41) and (42) are two equations in two unknowns for E^*/E and E'/E. Solving Eq. (41) for E' and inserting the result into Eq. (42) gives

$$E - E^* = \frac{n\cos\theta'}{n'\cos\theta}(E + E^*). \tag{43}$$

Using Snell's law, Eq. (34), to rewrite n/n' in terms of sine functions, and with the trig identity $\sin\alpha\cos\alpha = \frac{1}{2}\sin(2\alpha)$, Eq. (43) may be written

$$E - E^* = \frac{\sin(2\theta')}{\sin(2\theta)}(E + E^*) \tag{44}$$

or

$$\frac{E^*}{E} = \frac{\sin(2\theta) - \sin(2\theta')}{\sin(2\theta) + \sin(2\theta')}. \tag{45}$$

With the assistance of the trig identities

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{46}$$

and

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \tag{47}$$

Eq. (45) becomes

$$\frac{E^*}{E} = \frac{\tan(\theta-\theta')}{\tan(\theta+\theta')}. \tag{48}$$

Now return to Eq. (41) and use Snell's law to write it as

$$E' = \frac{\sin\theta'}{\sin\theta}(E + E^*). \tag{49}$$

Insert E^* from Eq. (48) to derive, with the help of the trig identity

$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}, \tag{50}$$

the result

$$\frac{E'}{E} = \frac{2\cos\theta\sin\theta'}{\sin(\theta+\theta')\cos(\theta-\theta')}. \tag{51}$$

Equations (48) and (51) are the Fresnel equations for Case 2 (**B** tangent to the boundary).

Alternative Derivations

Since we have four applicable boundary conditions, Eqs. (17)-(20), you may ask why we used only half of them—the continuity of the tangential components—in deriving both sets of Fresnel equations. The answer is that the Fresnel equations can alternatively be derived, producing identical or equivalent results, using other pairs of the four boundary condition equations. For instance, let us return to Case 2 and recall Eq. (42) for continuity in the tangential component of **E**, repeated here as

$$E - E^* = \frac{\cos\theta'}{\cos\theta}E'. \tag{52}$$

Instead of using continuity in the tangential component of $\mathbf{H}$, let us use continuity in the normal component of $\mathbf{D}$. The boundary condition $\Delta D_n = 0$ along with $\mathbf{D} = \epsilon\mathbf{E}$ gives

$$\epsilon_1(E + E^*)\sin\theta = \epsilon_2 E'\sin\theta' \tag{53}$$

or

$$E + E^* = \frac{\epsilon_2}{\epsilon_1}\frac{\sin\theta'}{\sin\theta}E'. \tag{54}$$

Referring back to Snell's law, Eq. (34), and $n = c\sqrt{\mu\epsilon}$, Eq. (12), it follows that

$$\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2} = \left(\frac{\sin\theta'}{\sin\theta}\right)^2, \tag{55}$$

and assuming $\mu_1 = \mu_2$, Eq. (55) turns Eq. (54) into

$$E + E' = \frac{\sin\theta}{\sin\theta'}E'. \tag{56}$$

Now Eqs. (52) and (56) are two equations in two unknowns for E^*/E and E'/E. After some more algebra that includes use of trig identities, we find

$$\frac{E^*}{E} = \frac{\tan(\theta-\theta')}{\tan(\theta+\theta')} \tag{57}$$

and

$$\frac{E'}{E} = \frac{2\sin\theta'\cos\theta}{\sin(\theta+\theta')\cos(\theta-\theta')}, \tag{58}$$

as before. The boundary conditions are quite robust!

References

1. See David Griffith, *Introduction to Electrodynamics*, 2nd ed. (Prentice-Hall, 1989), Ch. 4 for a discussion of electric polarization, Ch. 6 for magnetic polarization, and Ch. 7 for a detailed discussion.
2. See David Griffith, *Introduction to Electrodynamics*, 2nd ed. (Prentice-Hall, 1989), Ch. 4 for a discussion of electric polarization, and Ch. 6 for magnetic polarization.
3. If the medium is not homogeneous, then ϵ and μ will vary with position; if the medium is not isotropic, then the permittivity and permeability coefficients become components of second-rank tensors.
4. Attenuation may be neglected since we examine the amplitudes at the surface; only when the wave has penetrated some distance into a medium will attenuation, if present, be measurable.
5. The Compton effect, which uses special relativity and quantum mechanics to model the collision between a photon and free particle of mass m that is initially stationary, shows a shift in the light's wavelength (and thus its frequency) from λ to λ' according to $\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$, where h is Planck's constant, c the speed of light in vacuum, and θ the angle of the outgoing photon relative to the incoming one. Because h/c is so very small, the mass m must also be very small (e.g., an electron) for $\lambda' - \lambda$ to be appreciable. The light that bounces off the mirror is "Compton-scattering" off the entire mirror, whose shiny surface consists of electrons bound to their atoms. Thus, $\lambda' - \lambda \approx 0$ and the Compton effect goes over to the classical limit. The Compton effect (Arthur H. Compton, Phys. Rev. **22**, p. 411, 1923) was one of the crucial predictions confirmed by experiment that compelled the physics community to take relativity and quantum mechanics seriously.